On some results of Alladi and others

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This paper is dedicated to Krishna Alladi on the occasion of his 60^{th} birthday.

Abstract This paper discusses some previous results of Alladi and others, and considers generalizations and extensions of these results.

Key words: Arithmetic functions, Ramanujan's mock theta functions

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1 Introduction

In 1957, Alladi and Ramanujan [1] introduced the arithmetic function

$$A(n) := \sum_{d|n} 2^{\mu(d)}. \tag{1.1}$$

They proved that $A(n)$ is odd if an only if n is a prime power. The main goal of this paper is prove the following

Theorem 1.1.

$$A(n) = O\left((\log n)^2\right). \tag{1.2}$$

Theorem 1.1 has the corollary.

Corollary 1.2. *There is a constant C such that*

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$$\frac{A(n)}{(\log n)^2} \le C,$$

for $n \ge 2$.

2 On a generalization of the Alladi-Ramanujan function

Definition 2.1. *For* $k \ge 1$ *define*

$$A_k(n) := \sum_{d|n} k^{\mu(d)}. \tag{2.1}$$

Remark 2.2. Observe that $A(n) = A_2(n)$, where $A(n)$ was defined in (1.1).

Lemma 2.3. *Let* n *be a positive integer. Then*

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Proof. The result follows easily by induction on n.

Before thinking about equation (2.1) or continuing please examine Figure 1 below.

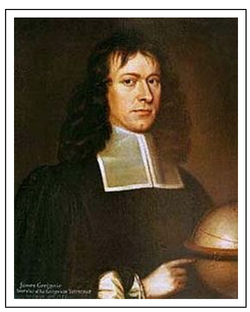

Fig. 1: James Gregory (1638–1675)

Then a straightforward calculation gives

Proposition 2.4. *Suppose* $p > 3$ *is prime and* $0 \le m \le p-1$.

(i) For $m = 0$ *or* $\left(\frac{-24m}{p}\right) = -1$ *we have*

$$\mathcal{K}_{p,m}(z) = q^{m/p} \prod_{n=1}^{\infty}(1-q^{pn}) \sum_{n=\lceil \frac{1}{p}(s_p-m)\rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k,p,pn+m-s_p)\,\zeta_p^k\right) q^n, \tag{2.2}$$

where $s_p = \frac{1}{24}(p^2 - 1)$, *and* $q = \exp(2\pi i z)$.

(ii) If $\left(\frac{-24m}{p}\right) = 1$ *we choose* $1 \le a \le \frac{1}{2}(p-1)$ *so that*

$$-24m \equiv (6a)^2 \pmod{p},$$

and we have

$$\begin{aligned} &\mathcal{K}_{p,m}(z) \qquad (2.3) \\ &= q^{m/p} \prod_{n=1}^{\infty}(1-q^{pn}) \left(\sum_{n=\lceil \frac{1}{p}(s_p - m) \rceil}^{\infty} \left(\sum_{k=0}^{p-1} N(k,p,pn+m-s_p)\,\zeta_p^k \right) q^n \right. \\ &\quad - \chi_{12}(p)\,(-1)^a \left(\zeta_p^{3a+\frac{1}{2}(p+1)} + \zeta_p^{-3a-\frac{1}{2}(p+1)} - \zeta_p^{3a+\frac{1}{2}(p-1)} - \zeta_p^{-3a+\frac{1}{2}(p-1)} \right) \\ &\qquad \left. \times\, q^{\frac{1}{p}(\frac{a}{2}(p-3a)-m)}\, \mathit{\Phi}_{p,a}(q) \right). \end{aligned}$$

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References

1. K. Alladi and S. Ramanujan, *On a certain arithmetic function*, Journal of Exceptional Mathematics, Vol. 101, No. 3, (1958), 1–15.