Combinatorial aspects of the theory of q-series

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Declaration

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree.

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This thesis is dedicated to the memory of Colin and Diana Carothers and Leslie and Muriel Hammond, my grandparents.

Abstract

This thesis is concerned mainly with the interplay between identities involving power series (which are called q-series) and combinatorics, in particular the theory of partitions. The thesis includes new proofs of some q-series identities and some ideas about the generating functions for the rank and crank, a new proof of the triple product identity and a combinatorial proof of a q-elliptic identity.

Contents

Chapter 1

Introduction

1.1 Partitions

1.1.1 Partitions and sets of partitions

A partition is a sequence of positive integers, for example each of the four sequences $(4,4,1)$, $(8,5,3,2)$, $(6,6,6)$ and (1) is a partition. Repetition, as in the first and third of these four sequences, is allowed. What is not permitted is for the sequence to be increasing, for instance $(1,4,4)$ is not a partition, nor is $(4,1,4)$. The Greek letter λ is used to denote a partition, if $\lambda = (4,4,1)$ then $\lambda_1 = 4$, $\lambda_2 = 4$ and $\lambda_3 = 1$.

Thus a *partition* λ is defined as being a finite, nonincreasing sequence of positive integers: $\lambda = (\lambda_1, \lambda_2, .., \lambda_k)$, k is therefore (the letter used to denote) the number of parts, which can also be written as $k = \#(\lambda)$, some authors use $|\lambda|$ for the number of parts. The partition λ is said to be composed of, or consist of, the parts, or entries, λ_i (where $1 \leq i \leq k$). The *weight* of a partition, $wt(\lambda)$, is defined to be the sum of the parts of λ, some authors use $||\lambda||$ for the weight, or we may say simply "a partition of n" instead of "a partition of weight n" (this may also be written as $\lambda \vdash n$). For example $\lambda = (4,4,1)$ is a partition of 9 having three parts. The empty partition, $\emptyset = (.)$, has weight zero and is composed of zero parts.

The set of all partitions, $\mathcal{P}$, occurs frequently in the thesis and so do certain subsets of it. Of particular interest are $\mathcal{D}$, $\mathcal{D}_e$, $\mathcal{D}_o$ and $\mathcal{O}$, where

$\mathcal{D} := \{\lambda \in \mathcal{P} : \lambda_i > \lambda_{i+1}\}$, the set of partitions into distinct parts,

$\mathcal{D}_e := \{\lambda \in \mathcal{D} : \#(\lambda) \equiv 0 \bmod 2\}$, the set of partitions into an even number of distinct parts,

$\mathcal{D}_o := \{\lambda \in \mathcal{D} : \#(\lambda) \equiv 1 \bmod 2\}$, the set of partitions into an odd number of distinct parts,

$\mathcal{O} := \{\lambda \in \mathcal{P} : \lambda_i \equiv 1 \bmod 2\}$, the set of partitions into odd parts.

The empty partition is an element of each of the sets $\mathcal{P}$, $\mathcal{D}$, $\mathcal{D}_e$ and $\mathcal{O}$ but is not in $\mathcal{D}_o$.

If H and H' are two sets of partitions and $\phi : H \to H'$ then the map ϕ is said to be *weight preserving* if $wt(\phi(\lambda)) = wt(\lambda)$, for all $\lambda \in H$. In this thesis any bijection between two sets of partitions is assumed to be weight preserving, unless it is stated otherwise.

Whenever H, say, is a set of partitions and there are m sets such that H is the disjoint union of the H_i then $H = H_1 \cup H_2 \cup ... \cup H_m$ will be called a *decomposition* of H (and where the union is not disjoint the word decomposition will not be used). Thus, for example, the set of partitions into distinct parts can be decomposed into $\mathcal{D} = \mathcal{D}_e \cup \mathcal{D}_o$. For any $H \subseteq \mathcal{P}$, the number of partitions in H of weight n will be written as $p(H, n)$. For $n \notin \mathbb{N}$, the convention is $p(H, n) := 0$ (the set $\mathbb{N}$ is understood to include 0, also used is the notation $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$). The total number of partitions of weight n is written simply as $p(n)$, or $p_{-1}(n)$, as is explained in section 1.2.3. The function $p(n)$ is called the *partition function.*

The hat sign, $\hat{}$, is used to indicate that an entry has been omitted. Thus $(5, 5, 3, \hat{3}, 2) = (5, 5, 3, 2)$, $(\hat{7}) = \emptyset$, $(\lambda_1, \lambda_2, ..., \hat{\lambda}_i, ..., \lambda_k)$ is λ with λ_i omitted and $(\lambda_1, \lambda_2, ..., \hat{\lambda}_i, \hat{...}, \hat{\lambda}_j, ..., \lambda_k)$ is λ with parts λ_i to λ_j omitted (which could, of course, be written as $(\lambda_1, \lambda_2, ..., \lambda_{i-1}, \lambda_{j+1}, ..., \lambda_k)$, but the hat is used to emphasise the omission).

1.1.2 The graph of a partition

Any nonempty partition can be visualised as the set of rows of coordinates in the bottom right quadrant of the plane where the ith row contains λ_i entries. This is called the *graphical representation* or the *Ferrers graph* (or simply the *graph*), $\mathcal{G}_\lambda$, of the partition. Before describing this explicitly, an example will help to illustrate the idea: If $\lambda = (4, 4, 1)$ then $\mathcal{G}_\lambda$ is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & & & \end{array}$$

(in order to be completely unambiguous it would be necessary to distinguish between the set $\{(1, -1), (2, -1), (3, -1), (4, -1), (1, -2), (2, -2), (3, -2), (4, -2), (1, -3)\}$ and the diagram above, but it is clear what is happening).

Now, given a nonempty partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$, for each i in the range $0 < i \leq \lambda_1$ let $\delta_\lambda(i)$ be the maximum l for which $\lambda_l - i \geq 0$. The formal definition is that the graph is the set of dots with integer coordinates (i, j) in the plane such that $(i, j) \in \mathcal{G}_\lambda$ if and only if $0 < i \leq \lambda_1$ and $-\delta_\lambda(i) \leq j < 0$.

The *dual,* or *conjugate,* of a partition is the partition whose graph is the one obtained by rotating the graph of the original partition around the falling diagonal (the set $\{(1, -1), (2, -2), ...\}$). Visually, the dual is obtained by interchanging each row with its corresponding column in $\mathcal{G}_\lambda$. In terms of the set, $\mathcal{G}_{\lambda^*} := \{\ (i, -j) : (j, -i) \in \mathcal{G}_\lambda\}$. The dual of λ will be denoted by λ'. In the case $\lambda = (4, 4, 1)$ the dual is $\lambda' = (3, 2, 2, 2)$. It follows immediately that the dual induces a weight preserving involution on $\mathcal{P}$, $wt(\lambda') = wt(\lambda)$ and $\lambda'' = \lambda$ for any $\lambda \in \mathcal{P}$. If λ is such that $\lambda' = \lambda$ then λ is said to be a *self-dual,* or *self-conjugate,* partition. Thus, for example, $(4, 4, 1)$ is not self-dual but $(4, 1, 1, 1)$ is.

1.2 Power Series

1.2.1 Generating functions and notation

Given a sequence $\{a_n\}$, it is often desirable to find an expression, $F(q)$, whose coefficients are the elements of the sequence, i.e. $F(q) = a_0 + a_1 q + ... + a_i q^i + ...$ When this happens the q-series $F(q)$ is said to be the *generating function* for the sequence $\{a_n\}$. For example, suppose $a_n = 1$ whenever $0 \leq n < 100$ and $a_n = 0$ for all other n. The generating for this sequence is $(1 - q^{100})/(1 - q)$. A power series in the parameter q is called a *q-series.*

The following standard notation will be used frequently throughout:

For z and $q \in \mathbb{C}$, define

$$(z; q)_n := \prod_{i=1}^{n} (1 - zq^{i-1}), \quad n \in \mathbb{N}^*$$

and

$$(z; q)_\infty := \prod_{i=1}^{\infty} (1 - zq^{i-1})$$

(the condition $|q| < 1$ is assumed in a nonterminating power series, this ensures convergence). In this thesis, the subscript n is assumed to be a nonnegative integer, which need not actually be the case.

The identity

$$(z;q)_n = \frac{(z;q)_\infty}{(zq^n;q)_\infty} \tag{1.1}$$

clearly holds for any $n \in \mathbb{N}^*$, and leads by extension to a definition of $(z;q)_n$ for any real n. In particular, $(z;q)_0 = 1$. For $m \in \mathbb{Z}$, the sequence $p_m(n)$ is defined by

$$\sum_{n\in\mathbb{Z}} p_m(n)q^n := (q;q)_\infty^m. \tag{1.2}$$

It follows that, for any m, $p_m(n) = 0$ whenever $n \notin \mathbb{N}$ (in fact the subscript m need not even be an integer, but in practice it always is). The product $1/(q;q)_\infty$ is sometimes written as $P(q)$ or simply as P (which should not be confused with the set $\mathcal{P}$).

The square bracket notation is defined, when $z \neq 0$, as

$$[z;q] := (z;q)_\infty(z^{-1}q;q)_\infty.$$

It is elementary, provided that $z \neq 0$, that

$$[z^{-1};q] = -z^{-1}[z;q] \tag{1.3}$$

and that

$$[zq;q] = -z^{-1}[z;q]. \tag{1.4}$$

Now, an expression in z and q (such as $(z;q)_n$, for some given n) can be seen as a power series in q, whose coefficients are expressions in z, for example,

$$(z;q)_3 = (1-z) + (-z+z^2)q + (-z+z^2)q^2 + (z^2-z^3)q^3$$

and it can also be seen as a power series in z whose coefficients are expressions in q,

$$(z;q)_3 = 1-(1+q+q^2)z + (q+q^2+q^3)z^2 - q^3z^3.$$

This suggests it might prove helpful to find a general expression for $(z;q)_n$ as a power series in z, so the question is, what is the coefficient of z^i in $(z;q)_n$? With this in mind it is now helpful to introduce the q-binomial coefficients.

1.2.2 The q-binomial coefficients

The q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_i(q;q)_{n-i}}. \tag{1.5}$$

The subscript q can be dropped safely from the expression on the left, except in chapter 3 where the subscript is q^2. The q-binomial coefficients are introduced at this stage because they appear in the expression for $(z;q)_n$, but before addressing that a few properties of the q-binomial coefficients are worth mentioning: Clearly,

$$\begin{aligned} \begin{bmatrix} n \\ i \end{bmatrix}_q &= \frac{(q;q)_n}{(q;q)_i(q;q)_{n-i}} \\ &= \frac{(q;q)_{n-i}(q^{n-i+1};q)_i}{(q;q)_i(q;q)_{n-i}} \\ \Rightarrow \begin{bmatrix} n \\ i \end{bmatrix} &= \frac{(q^{n-i+1};q)_i}{(q;q)_i}. \end{aligned} \tag{1.6}$$

Now, it can easily be shown that

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}. \tag{1.7}$$

This is (3.3.3) in [2], and it follows from this identity that, when i and n are both nonnegative integers and $0 \le i \le n$, $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]$ is in fact a polynomial, of degree $ni - i^2$. The q-binomial coefficients are also called Gaussian polynomials. Note that the identity $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ n-i \end{smallmatrix}\right]$ follows from the definition (1.5) of the q-binomial coefficients.

Closely related to the role the q-binomial coefficients play in the expansions for $(z;q)_n$ and $(z;q)_n^{-1}$, that is in identities (1.9) and (1.10) below, is the fact that they are the generating functions for partitions where both the largest part and the number of parts are forbidden to exceed a given pair of nonnegative integer values, a and b, say. This is dealt with in the following

Lemma:

Let $\mathcal{U}(a,b) := \{\lambda : \lambda_1 \le a, \#(\lambda) \le b\}$, thus $\mathcal{U}(a,b)$ is the set of all partitions whose graph lies inside a rectangle of length a and height b. Then

$$\sum_{\lambda \in \mathcal{U}(a,b)} q^{wt(\lambda)} = \begin{bmatrix} a+b \\ a \end{bmatrix}. \tag{1.8}$$

Briefly, the proof involves the inductive step that firstly $\left[\begin{smallmatrix} a+b-1 \\ a \end{smallmatrix}\right]$ should be the generating function for partitions having less than b parts, none of which exceed a (i.e. partitions in $\mathcal{U}(a, b-1)$), and secondly $q^b\left[\begin{smallmatrix} a+b-1 \\ a-1 \end{smallmatrix}\right]$ should be the generating function for partitions into precisely b parts, no part exceeding a (clearly (1.8) holds whenever either a or b is 0). This is explained in (3.4.1.) in [2] and also in [16].

It follows from (1.8) that the q-binomial coefficients are symmetric, in the sense that if $\begin{bmatrix} n \\ i \end{bmatrix} = a(0) + a(1)q + ... + a(ni - i^2)q^{ni-i^2}$ then $a(k) = a(ni - i^2 - k)$. It has also been shown, in [16], that the they are unimodal, which is to say that if $k \le \frac{ni-i^2}{2}$ then $a(k-1) \le a(k)$.

Furthermore,

$$\lim_{q \to 1} \begin{bmatrix} n \\ i \end{bmatrix} = \binom{n}{i},$$

the familiar binomial coefficient, as maybe seen by applying l'Hopital's rule to (1.6).

The following identity, due to Euler, states that the coefficient of z^i in $(z;q)_n$ is $(-1)^i \begin{bmatrix} n \\ i \end{bmatrix} q^{(i^2-i)/2}$,

$$(z;q)_n = \sum_{i=0}^{n} (-1)^i z^i \begin{bmatrix} n \\ i \end{bmatrix} q^{(i^2-i)/2} \tag{1.9}$$

and the coefficient of z^i in the reciprocal is given by a result of Rothe,

$$\frac{1}{(z;q)_n} = \sum_{i=0}^{\infty} z^i \begin{bmatrix} n+i-1 \\ i \end{bmatrix}. \tag{1.10}$$

For a proof of the above two identities, see Theorem 3.3 in [2], or see [3] where they are derived from the q-binomial theorem.

Now, since $(q;q)_n$ can be defined for any real n, it follows that $\begin{bmatrix} n \\ i \end{bmatrix}$ can be defined for i and n any real numbers. In fact the sums on the right in (1.9) and (1.10) can be replaced by sums with i ranging over all integers, for if n is a nonnegative integer and $i \in \mathbb{Z} \setminus \{0, 1, ..., n\}$ then a limiting argument applied to (1.5) and (1.1) gives $\begin{bmatrix} n \\ i \end{bmatrix} = 0$ (which is to be expected, as $\binom{n}{i} = 0$ for all such i).

The limiting case as $q \to 1$ in identity (1.9) is the familiar expansion for $(1-z)^n$ and likewise, $q \to 1$ in identity (1.10) gives the expansion for $(1-z)^{-n}$. This illustrates a general principle, the limit as $q \to 1$ in any identity involving q-binomial coefficients is an identity involving ordinary binomial coefficients.

1.2.3 Partitions and q-series

It is now possible to give an outline of the connection between q-series and partitions. For instance, if H is the set $\{\emptyset, (1), (1,1), ...\} \subset \mathcal{P}$ of partitions with each part equal to 1 then

$$\begin{aligned} \frac{1}{1-q} &= 1 + q + q^2 + q^3 + ... \\ &= 1 + q^1 + q^{1+1} + q^{1+1+1} + ... \\ &= \sum_{\lambda \in H} q^{wt(\lambda)}. \end{aligned}$$

What this means is that $(1-q)^{-1}$ is the generating function for $p(H, n)$,

$$\sum_{n \ge 0} p(H,n) q^n = (1-q)^{-1}.$$

More generally, if H were to be the set of all partitions composed of elements from a given subset $\{\alpha_1, \alpha_2, ..., \alpha_l\} \subseteq \mathbb{N}^*$, then it would follow that

$$\prod_{i=1}^{l} \left(\frac{1}{1-q^{\alpha_i}} \right) = \sum_{\lambda \in H} q^{wt(\lambda)}$$

which is to say that

$$\sum_{n \geq 0} p(H, n) q^n = \prod_{i=1}^{l} (1-q^{\alpha_i})^{-1}.$$

This includes the case H is the whole of $\mathcal{P}$ (as can be seen by taking $l = \infty$ and, for example, $\alpha_i = i$) so

$$\prod_{i=1}^{\infty} \frac{1}{(1-q^i)} = \sum_{\lambda \in \mathcal{P}} q^{wt(\lambda)}$$

which is to say that

$$\sum_{n \geq 0} p(n) q^n = (q;q)_\infty^{-1}. \tag{1.11}$$

Putting $m = -1$ in (1.2) gives $p(n) = p_{-1}(n)$. In fact, $p_{-m}(n)$ is the number of ordered m-tuples of partitions,

$$\frac{1}{(q;q)_\infty^m} = \sum_{\lambda \in \mathcal{P}^m} q^{wt(\lambda)}.$$

In particular, if $m = -2$ then $p_{-2}(n)$ is the number of ordered pairs of partitions. The behaviour of $p_{-2}(n)$, in particular some of its congruences (and an explanation for these congruences, namely the birank) is the subject of the second half of chapter 5.

Some subsets of $\mathcal{P}$ have already been defined. Concerning these, it is not hard to show that

$$\prod_{i=1}^{\infty} \frac{1}{(1-q^{2i-1})} = \sum_{\lambda \in \mathcal{O}} q^{wt(\lambda)}$$

which is to say that

$$\sum_{n \geq 0} p(\mathcal{O}, n) q^n = (q;q^2)_\infty^{-1} \tag{1.12}$$

and

$$\prod_{i=1}^{\infty} (1+q^i) = \sum_{\lambda \in \mathcal{D}} q^{wt(\lambda)}$$

which is to say that

$$\sum_{n \geq 0} p(\mathcal{D}, n) q^n = (-q;q)_\infty \tag{1.13}$$

and finally, $p_1(n) = p(\mathcal{D}_e, n) - p(\mathcal{D}_o, n)$, or equivalently,

$$\sum_{n \geq 0} (p(\mathcal{D}_e, n) - p(\mathcal{D}_o, n)) q^n = (q;q)_\infty. \tag{1.14}$$

So $(q;q)_\infty$ maybe called, with a slight abuse of terminology, the generating function for partitions into distinct parts where those having an odd number of parts are counted negatively.

1.3 A bit of history

1.3.1 Elementary Identities

In the mid eighteenth cenury Euler observed that

$$\prod_{i=1}^{\infty}(1-q^i) = 1-q-q^2+q^5+q^7-q^{12}-q^{15}+....$$

The product $\prod_{i=1}^{\infty}(1-q^i)$ is sometimes called the Euler product (the notation $(q;q)_\infty$ is a twentieth century development).

He went on to demonstrate that

$$\prod_{i=1}^{\infty}(1-q^i) = \sum_{m\in\mathbf{Z}}(-1)^m q^{\frac{m(3m+1)}{2}}. \tag{1.15}$$

This is called the pentagonal number theorem, because a number n is said to be pentagonal if and only if $n = \frac{m(3m+1)}{2}$ for some integer m. Euler was probably aware that $(q;q)_\infty$ is the generating function for $p(\mathcal{D}_e, n) - p(\mathcal{D}_o, n)$ but there is no evidence to suggest that he considered looking for a combinatorial proof that

$$p(\mathcal{D}_e, n) - p(\mathcal{D}_o, n) = \begin{cases} (-1)^m & \text{if } n = \frac{m(3m+1)}{2} \\ 0 & \text{if } n \text{ is not pentagonal} \end{cases} \tag{1.16}$$

and Franklin was the first to find one (Legendre apparently suggested that there might be a combinatorial proof, though he did not give a proof himself). Franklin's proof is described in section 2.1. Briefly though, Franklin's proof involves decomposing $\mathcal{D}$ into the union of three sets. Between the first two sets there exists a bijection (or at least a bijection can be defined) and the third set contains no partitions of weight n unless $n = \frac{m(3m+1)}{2}$ for some integer m, whence it contains exactly one partition of weight n (which has an even number of parts precisely when m is even, as will be explained).

Franklin's proof, as outlined above, involves pairing off elements of a given set and counting those that remain is an instance of a proof by involution. In chapter 4 a new proof of the Jacobi triple product identity is presented. This proof also involves an involution. The triple product identity states

$$[z;q](q;q)_\infty = \sum_{n\in\mathbf{Z}}(-1)^n z^n q^{\frac{n^2-n}{2}}. \tag{1.17}$$

This is not to say that involutions are the only approach to q-series identities. For example given certain identities it is possible to find a bijection from the set whose generating function is the expression on the left to the set with generating function the set on the right, as opposed to an involution which 'stays on one side of the equation'.

As an illustration of this, consider

$$
\begin{aligned}
(-q;q)_\infty &= \prod_{i=1}^{\infty}(1+q^i) \\
&= \prod_{i=1}^{\infty}\frac{(1-q^{2i})}{(1-q^i)} \\
&= \prod_{i=1}^{\infty}\frac{(1-q^{2i})}{(1-q^{2i-1})(1-q^{2i})} \\
&= \prod_{i=1}^{\infty}\frac{1}{(1-q^{2i-1})}
\end{aligned}
$$

so

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}. \tag{1.18}$$

Thus by (1.13) and (1.12), $p(\mathcal{D},n) = p(\mathcal{O},n)$. This suggests that there might a weight preserving bijection between the set of partitions into distinct parts and the set of partitions into odd parts. In fact, there is more than one. One of these, the bijection that was found by Sylvester, is described in section 2.2 and is related to the bijection in section 3.1.

Now, having established that $p(\mathcal{D},n) = p(\mathcal{O},n)$, it is clear that $p(\mathcal{O},n) \equiv 0 \bmod 2$ if and only if n is not a pentagonal number (by virtue of (1.16), and the obvious fact that $p(\mathcal{D}_e,n) - p(\mathcal{D}_o,n) \equiv p(\mathcal{D},n) \bmod 2$). Hence it seems reasonable to ask if the set $\mathcal{O}$ can be decomposed into two sets in such a way as to ensure that the number of elements of $\mathcal{O}$ (i.e. partitions, into odd parts) of weight n in one of the sets is the same as the number of elements of weight n in the other set, unless n is a pentagonal number (just as $\mathcal{D}$ can be decomposed into two such sets, $\mathcal{D}_e$ and $\mathcal{D}_o$, which is where the identity (1.16) comes from).

The question is then, what property of partitions in $\mathcal{O}$ is it that 'almost half' of them have that could give rise to an involution on $\mathcal{O}$? The Franklin involution reverses mod 2 the parity of the number of parts of a partition in $\mathcal{D}$. As, clearly, the first part of any nontrivial partition in $\mathcal{O}$ has first part congruent to 1 mod 2, the proposed involution on $\mathcal{O}$ clearly will not have this effect. The Franklin involution on $\mathcal{D}$ also has the effect of reversing the parity mod 2 of the number of parts of any partition in $\mathcal{D}$ on which it acts. This cannot be the case for the involution on $\mathcal{O}$, as for any partition $\lambda \in \mathcal{O}$, the number of parts in the partition has the same parity as the weight of the partition: $\#(\lambda) \equiv wt(\lambda) \bmod 2$.

Instead it transpires that the place to look is the residue of the first part, of a partition in $\mathcal{O}$, mod 4. This to say that an involution on $\mathcal{O}$ is presented that has the effect of taking partitions with $\lambda_1 \equiv 1 \bmod 4$ to those with $\lambda_1 \equiv 3 \bmod 4$, and vice versa. This is done in section 3.1.

1.3.2 Congruences in the Euler product

In 1918 the Indian mathematican Ramanujan wrote a letter to Hardy. The letter included, amongst other things, the following congruences in the partition function which he (Ramanujan) had conjectured,

$$p(5n+4) \equiv 0 \bmod 5, \tag{1.19}$$

$$p(7n+5) \equiv 0 \bmod 7, \tag{1.20}$$

$$p(11n+6) \equiv 0 \bmod 11, \tag{1.21}$$

of which he later gave proofs, which can be found in [29].

The first two of these are entirely elementary, the following is a sketch of a proof of (1.19): Dividing both sides of the triple product identity (1.17) by $1-z$ gives

$$\frac{[z;q]}{1-z}(q;q)_\infty = \frac{1}{1-z}\sum_{n\in\mathbb{Z}}(-1)^n z^n q^{\frac{n^2-n}{2}}$$

$$= \frac{1-z}{1-z} - \frac{(z^{-1}-z^2)}{1-z}q + \frac{(z^{-2}-z^3)}{1-z}q^3 - \frac{(z^{-3}-z^4)}{1-z}q^6 + \ldots$$

which is equal to

$$1-(z+1+z^{-1})q+(z^2+z+1+z^{-1}+z^{-2})q^3-(z^3+z^2+z+1+z^{-1}+z^{-2}+z^{-3})q^6+\ldots$$

and it follows that

$$(zq;q)_\infty(z^{-1}q;q)_\infty(q;q)_\infty = \sum_{n\geq 0}(-1)^n(z^n+z^{n-1}+\ldots+z^{-n})q^{\frac{(n^2+n)}{2}} \tag{1.22}$$

and then setting $z=1$ in (1.22) gives,

$$(q;q)_\infty^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}. \tag{1.23}$$

Two power series, $a_0+a_1q+a_2q^2+\ldots$ and $b_0+b_1q+b_2q^2+\ldots$, are said to be congruent modulo r if and only if $a_i \equiv b_i \bmod r$ for all i.

Now for any prime r, $(1-q)^r \equiv 1-q^r \bmod r$ by the binomial theorem, so for $i=1,2,\ldots$, $(1-q^i)^r \equiv (1-q^{ir}) \bmod r$. Taking products over all i in the positive integers gives

$$r \text{ is prime} \Rightarrow (q;q)_\infty^r \equiv (q^r;q^r)_\infty \bmod r. \tag{1.24}$$

The generating function for $p(n)$,

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty}$$

together with (1.24) gives,

$$\sum_{n\geq 0} p(n)q^n \equiv \frac{1}{(q;q)_\infty} \frac{(q;q)_\infty^5}{(q^5;q^5)_\infty} \bmod 5$$

so

$$\sum_{n\geq 0} p(n)q^n \equiv \frac{(q;q)_\infty^3 (q;q)_\infty}{(q^5;q^5)_\infty} \bmod 5$$

and it follows, by (1.23) and (1.15), that

$$\sum_{n\geq 0} p(n)q^n \equiv \frac{\left(\alpha_0(q^5) + q\alpha_1(q^5)\right)\left(\beta_0(q^5) + q\beta_1(q^5) + q^2\beta_2(q^5)\right)}{(q^5;q^5)} \bmod 5$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1$ and β_2 are five particular power series (which can of course be determined explicitly, but it's not necessary to actually do this). The point is that if $t(n)$, say, is the coefficient in the above expression then clearly $t(5n+4) \equiv 0 \bmod 5$.

Not only did Ramanujan conjecture the congruences (1.19), (1.20) and (1.21), he also made conjectures for higher powers of 5, 7 and 11. He conjectured that

$$x = 5 \text{ or } 11 \Rightarrow \quad p(n) \equiv 0 \bmod x^m, \text{ if } 24n \equiv 1 \bmod x^m. \tag{1.25}$$

Ramanujan was correct to conjecture (1.25). In fact he conjectured that (1.25) was true for $x = 5$, 7 and 11, which is not actually the case. Chowla noticed that whilst $p(243)$ is divisible by 49, it is not divisible by 343 (so (1.25) fails for $m = 3$, if $x = 7$).

The correct statement for powers of 7 is

$$p(n) \equiv 0 \bmod 7^{\lfloor m/2 \rfloor + 1}, \text{ if } 24n \equiv 1 \bmod 7^m. \tag{1.26}$$

The brackets here denote the floor function (the greatest integer no more than $m/2$), i.e. $\lfloor m/2 \rfloor$ is $m/2$ if m is even or $(m-1)/2$ if m is odd.

The corrected version of (1.25) was proved by Atkin, [4].

Chapter 2

Franklin and Sylvester

2.1 The Franklin bijection

This section outlines Franklin's combinatorial proof of (1.16)

$$p(\mathcal{D}_e, n) - p(\mathcal{D}_o, n) = \begin{cases} (-1)^m & \text{if } n = \frac{m(3m+1)}{2}, \\ 0 & \text{if } n \text{ is not pentagonal.} \end{cases}$$

Franklin's approach was to construct a function F defined on most partitions into distinct parts that satisfies the following conditions

$$F \text{ is weight preserving:} \quad wt(F(\lambda)) = wt(\lambda),$$

$$F \text{ reverses the parity of the number of parts:} \quad \#(F(\lambda)) \not\equiv \#(\lambda) \bmod 2,$$

$$F \text{ is an involution:} \quad F(F(\lambda)) = \lambda.$$

The reason for the word most should become clear shortly.

It is necessary to introduce some more (standard) notation before outlining his idea, which appears in 14.5 in [1], and as theorem 1.6 in [2], as theorem 19.5 in [6], in [7], in chapter III of [8], as 19.10 in [20], as theorem 15.5 in [25], as 256 in [26], as §100 in [27], in chapter 5 in [31] and in [12]. For a partition into distinct parts, the slope, $\sigma(\lambda)$ or simply σ, of the partition is the maximum value of i for which $\lambda_1 - \lambda_i = i - 1$. Also, for a partition into distinct parts, $s(\lambda)$, or simply s, is the smallest part of the partition. The σ and s here defined (on partitions into distinct parts) are analogous to (but not the same as) the Σ and S defined (on partitions into odd parts) in section 3.2 and to the σ and s defined (on partitions into nested parts) in subsection 3.4.2. The sets A and B are defined, provisionally at this stage, as

$$A = \{\lambda : \sigma(\lambda) < s(\lambda)\}$$

and

$$B = \{\lambda : \sigma(\lambda) \geq s(\lambda)\}.$$

It is now possible to tentatively define $F : \mathcal{D} \to \mathcal{D}$ as the map that removes the slope and places it as the smallest part if $\lambda \in A$, or if $\lambda \in B$ then F does the opposite, removes the smallest part and adds 1 to the appropriate number of entries at the start,

$$F(\lambda) := \begin{cases} (\lambda_1 - 1, ..., \lambda_\sigma - 1, \lambda_{\sigma+1}, ..., \lambda_{k-1}, \lambda_k, \sigma) & \text{if } \sigma(\lambda) < s(\lambda) \\ (\lambda_1 + 1, ..., \lambda_s + 1, \lambda_{s+1}, ..., \lambda_{k-1}, \hat{\lambda_k}) & \text{if } \sigma(\lambda) \geq s(\lambda) \end{cases}$$

where the hat, $\hat{\cdot}$, indicates omission.

Clearly F is weight preserving, and $F(\lambda)$ has either one part more than λ or one part less. Furthermore F reverses the inequality $\sigma(\lambda) < s(\lambda)$ (meaning $\sigma(\lambda) < s(\lambda)$ if and only if $\sigma(F(\lambda)) \geq s(F(\lambda))$) and from this it follows that $F(F(\lambda)) = \lambda$. Hence F satisfies the above conditions. But there are some 'exceptional' partitions on which it is impossible to define F, it is necessary to establish precisely which partitions these are.

For any nonempty partition in $\mathcal{D}$, the inequality $\sigma(\lambda) \leq \#(\lambda)$ will hold, since the slope of such a partition cannot exceed the number of parts in the partition. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is such that $\sigma(\lambda) < \#(\lambda)$, i.e. there exists some i such that $1 < i \leq k$ for which there is a strict inequality $\lambda_i < \lambda_{i-1} - 1$, then there is no problem in defining $F(\lambda)$ (although $F(\lambda)$ itself might be a partition having slope equal to number of parts). Thus it is only necessary to consider $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \in \mathcal{D}$ for which $\sigma(\lambda) = \#(\lambda)$, i.e. partitions of the form $\lambda = (\lambda_1, \lambda_1 - 1, \lambda_1 - 2, ..., \lambda_1 - k + 1)$. Any such partition satisfies either $k < \lambda_1 - k$, $k = \lambda_1 - k$, $k = \lambda_1 - k + 1$ or $k > \lambda_1 - k + 1$. These four cases are now considered:

<u>Case 1: $\sigma(\lambda) = \#(\lambda)$, $k < \lambda_1 - k$:</u> In this case $\lambda = (\lambda_1, \lambda_1 - 1, ..., \lambda_1 - k + 1)$, and so $s(\lambda) = \lambda_1 - k + 1$ and $\sigma(\lambda) = k$. Thus, since $\lambda_1 - k > k$, $s(\lambda) > \sigma(\lambda)$ and so $\lambda \in A$. Hence $F(\lambda) = (\lambda_1 - 1, \lambda_1 - 2, ..., \lambda_1 - k, k)$ is well defined, since $\lambda_1 - k > k$ ensures that the penultimate entry exceeds the last entry (and k, being the number of parts, is a positive integer implies that the last entry in $F(\lambda)$ is positive).

<u>Case 2: $\sigma(\lambda) = \#(\lambda)$, $k = \lambda_1 - k$:</u> As in case 1, $\lambda = (\lambda_1, \lambda_1 - 1, ..., \lambda_1 - k + 1)$, and so $s(\lambda) = \lambda_1 - k + 1$ and $\sigma(\lambda) = k$. Thus, since $\lambda_1 - k = k$ and $s(\lambda) > \sigma(\lambda)$, if $F(\lambda)$ were indeed defined, then λ would be an element of A. This would give $F(\lambda) = (\lambda_1 - 1, \lambda_1 - 2, ..., \lambda_1 - k, k)$ but, since $\lambda_1 - k = k$, $F(\lambda)$ is not permitted on account of the last two entries being equal. Hence, in this case, $F(\lambda)$ is not defined. Since these partitions are those having length k and $\lambda_1 = 2k$, they are of the form $\lambda = (2k, 2k - 1, ..., k + 1)$. Hence, if X is the set of partitions in $\mathcal{D}$ for which F is not defined then $\emptyset, (2), (4, 3), (6, 5, 4), ... \in X$. Note that this illustrates why the set A was initally defined only provisionally, there are some elements $\lambda \in \mathcal{D}$ for which $\sigma(\lambda) < s(\lambda)$ but for which the map $\lambda \to F(\lambda)$ can not be defined. Case 3 now demonstrates that the same is true for some of the partitions that should, it seems at first, belong in B.

Case 3: $\sigma(\lambda) = \#(\lambda)$, $k = \lambda_1 - k + 1$: Again $\lambda = (\lambda_1, \lambda_1 - 1, ..., \lambda_1 - k + 1)$, and so $s(\lambda) = \lambda_1 - k + 1$ and $\sigma(\lambda) = k$. Since $\lambda_1 - k + 1 = k$, it follows that $s(\lambda) = \sigma(\lambda)$ and so λ should be an element of B, provided that $F(\lambda)$ is indeed defined. This would imply that λ has length k and so $F(\lambda)$ should have length $k-1$, i.e. $F(\lambda) = (\lambda_1 + 1, \lambda_1, \lambda_1 - 1, ..., \lambda_1 - k + 2)$. Now $wt(\lambda) = \lambda_1 + (\lambda_1 - 1) + ... + (\lambda_1 - k + 1) = k\lambda_1 - \binom{k}{2}$ and similarly $wt(F(\lambda)) = (k-1)\lambda_1 + 1 - \binom{k-2}{2}$. But this gives $wt(\lambda) - wt(F(\lambda)) = \lambda_1 - 2k + 2 = 1$ (since $\lambda_1 = 2k - 1$) and so $wt(F(\lambda)) \neq wt(\lambda)$. Hence for these such partitions, F is not defined. These partitions have length k and $\lambda_1 = 2k - 1$, so are of the form $\lambda = (2k-1, 2k-2, ..., k)$. Hence $(1), (3,2), (5,4,3), ... \in X$.

Case 4: $\sigma(\lambda) = \#(\lambda)$, $k > \lambda_1 - k + 1$: Again $\lambda = (\lambda_1, \lambda_1 - 1, ..., \lambda_1 - k + 1)$, and so $s(\lambda) = \lambda_1 - k + 1$ and $\sigma(\lambda) = k$. Now $\lambda_1 - k + 1 < k$ and so $s(\lambda) < \sigma(\lambda)$, hence $\lambda \in B$. There is no problem here, as $F(\lambda) = (\lambda_1 + 1, \lambda_1, \lambda_1 - 1, ..., \lambda_1 - s + 2, \lambda_1 - s, ..., \lambda_1 - k + 2)$.

Thus no problems arise from cases 1 and 4, from cases 2 and 3 it is seen that the set of exceptional partitions on which F is not defined is $X = \{\emptyset, (1), (2), (3,2), (4,3), ...\}$. Clearly $\mathcal{D}$ is the disjoint union of the set of nonexceptional partitions and X, the set of exceptional partitions, and

$$X = \{(2k, 2k-1, ..., k+1) : k \geq 0\} \cup \{(2k+1, 2k-1, ..., k+1) : k \geq 0\}.$$

The set X can be split into X_e and X_o where $X_e := \mathcal{D}_e \cap X$ and $X_o := \mathcal{D}_o \cap X$. Note that $X_e = X_e^1 \cup X_e^2$ and $X_e = X_o^1 \cup X_o^2$ where

$$X_e^1 = \{(4k, 4k-1, ..., 2k+1) : k \geq 0\},$$

$$X_e^2 = \{(4k+3, 4k+2, ..., 2k+2) : k \geq 0\},$$

$$X_o^1 = \{(4k+2, 4k+1, ..., 2k+2) : k \geq 0\},$$

$$X_o^2 = \{(4k+1, 4k, ..., 2k+1) : k \geq 0\}.$$

It follows that

$$\sum_{\lambda \in X_e^1} q^{wt(\lambda)} = \sum_{k \geq 0} q^{6k^2+k} \tag{2.1}$$

(because $4k + 4k - 1 + ... + 2k + 1 = 6k^2 + k$) and

$$\sum_{\lambda \in X_e^2} q^{wt(\lambda)} = \sum_{k \geq 0} q^{6k^2+11k+5}, \tag{2.2}$$

$$\sum_{\lambda \in X_o^1} q^{wt(\lambda)} = \sum_{k \geq 0} q^{6k^2+7k+2}, \tag{2.3}$$

$$\sum_{\lambda \in X_o^2} q^{wt(\lambda)} = \sum_{k \geq 0} q^{6k^2+5k+1}. \tag{2.4}$$

The sets A and B having been provisionally defined above can now be properly defined,

$$A := \{\lambda : \sigma(\lambda) < s(\lambda) : \lambda \notin X\}$$

and

$$B := \{\lambda : \sigma(\lambda) \geq s(\lambda) : \lambda \notin X\}.$$

Having thus defined A and B, it is correct to say that $F : A \to B$ is a bijection.

It might seem unfortunate that establishing which set, A or B, a particular nonexceptional partition is an element of is not the same as establishing whether the number of parts in that partition is even or odd. Given that the motivation for the above was to prove (1.16), and yet what has actually been accomplished is the construction of a map that alters the relationship between the slope and the smallest part of the partition, what was the point? Looking at the map $\lambda \to F(\lambda)$ again, however, two observations can be made: Firstly it is clear that F reverses the parity of the first part of the partition, because the first part is either increased or decreased by 1, and secondly F reverses the paritiy of the number of parts, for again this number is either one more or one less in $F(\lambda)$ than in the original λ. The first of these observations, it will be shown leads to identity (2.5) below, but before that the train of thought in the second observation needs to be followed through.

The set A can be written as the disjoint union of A_0 and A_1 where

$$A_i := \{\lambda \in A : \#(\lambda) \equiv i \bmod 2\}$$

and similarly for B. Just as $F : A \to B$ is a bijection, so too is $F : (A_0 \cup B_0) \to (A_1 \cup B_1)$. As has been noted, $(-1)^{\#(\lambda)} \neq (-1)^{\#F(\lambda)}$ and since $\lambda \in A_0 \cup B_0 \Rightarrow (-1)^{\#(\lambda)} = +1$ and $\lambda \in A_1 \cup B_1 \Rightarrow (-1)^{\#(\lambda)} = -1$, it follows that to determine $p(\mathcal{D}_e, n) - p(\mathcal{D}_o, n)$ for some given n it suffices to consider only those partitions in $\mathcal{D}$ that are also in X (because any nonexceptional partition, λ, can be paired off with $F(\lambda)$ which has the same weight and opposite sign). Thus

$$\sum_{\lambda \in \mathcal{D}} (-1)^{\#(\lambda)} q^{wt(\lambda)} = \sum_{\lambda \in A_0 \cup B_0} (-1)^{\#(\lambda)} q^{wt(\lambda)} + \sum_{\lambda \in A_1 \cup B_1} (-1)^{\#(\lambda)} q^{wt(\lambda)} + \sum_{\lambda \in X} (-1)^{\#(\lambda)} q^{wt(\lambda)}$$

so

$$\sum_{\lambda \in \mathcal{D}} (-1)^{\#(\lambda)} q^{wt(\lambda)} = \sum_{\lambda \in A_0 \cup B_0} q^{wt(\lambda)} - \sum_{\lambda \in A_1 \cup B_1} q^{wt(\lambda)} + \sum_{\lambda \in X} (-1)^{\#(\lambda)} q^{wt(\lambda)}$$

and the expression reduces to a sum over X. Now the sum over X is equal to the (sum of the) sums over the four subsets,

$$\sum_{\lambda \in X} (-1)^{\#(\lambda)} q^{wt(\lambda)} = \sum_{\lambda \in X_e^1} q^{wt(\lambda)} + \sum_{\lambda \in X_e^2} q^{wt(\lambda)} - \sum_{\lambda \in X_o^1} q^{wt(\lambda)} - \sum_{\lambda \in X_o^2} q^{wt(\lambda)}$$

which, from (2.1), (2.2), (2.3) and (2.4) is equal to $1 - q - q^2 + q^5 + \ldots$. This is the combinatorial proof of (1.16).

Now, the first observation about F was that it reverses the parity of the first part. This means that, as well as thinking of F as a map from the set of nonexceptional partitions (in $\mathcal{D}$) having an even number of parts to those having an odd number of parts, it is equally valid to consider it as a map from the set of nonexceptional partitions having first part even to those having first part odd. So if $\mathcal{D}'_e$ denotes the set of partitions in $\mathcal{D}$ having first part even and $\mathcal{D}'_o$ the set of such partitions with first part odd, then

$$\sum_{\lambda\in\mathcal{D}}(-1)^{\lambda_1}q^{wt(\lambda)} = \sum_{\lambda\in\mathcal{D}'_e\setminus X}(-1)^{\lambda_1}q^{wt(\lambda)} + \sum_{\lambda\in\mathcal{D}'_o\setminus X}(-1)^{\lambda_1}q^{wt(\lambda)} + \sum_{\lambda\in X}(-1)^{\lambda_1}q^{wt(\lambda)}$$

so

$$\sum_{\lambda\in\mathcal{D}}(-1)^{\lambda_1}q^{wt(\lambda)} = \sum_{\lambda\in\mathcal{D}'_e\setminus X}q^{wt(\lambda)} - \sum_{\lambda\in\mathcal{D}'_o\setminus X}q^{wt(\lambda)} + \sum_{\lambda\in X}(-1)^{\lambda_1}q^{wt(\lambda)}$$

and, similarly to the above case, this becomes

$$\sum_{\lambda\in X_e^1}q^{wt(\lambda)} - \sum_{\lambda\in X_e^2}q^{wt(\lambda)} + \sum_{\lambda\in X_o^1}q^{wt(\lambda)} - \sum_{\lambda\in X_o^2}q^{wt(\lambda)} = 1 - q + q^2 - q^5 + \ldots$$

hence

$$p(\mathcal{D}'_e,n) - p(\mathcal{D}'_o,n) = \begin{cases} +1 & \text{if } n = \frac{m(3m+1)}{2} \text{ and } m \geq 0 \\ -1 & \text{if } n = \frac{m(3m+1)}{2} \text{ and } m < 0 \\ 0 & \text{if } n \text{ is not pentagonal} \end{cases} \tag{2.5}$$

2.1.1 The generating function for $p(\mathcal{D}'_e,n) - p(\mathcal{D}'_o,n)$

Let $\mathcal{D}(k)$ be the set of partitions into distinct parts with first part k. The generating function for $p(\mathcal{D}(k),n)$ is

$$\sum_{n\geq 0}p(\mathcal{D}(k),n)q^n = q^k(1+q^{k-1})(1+q^{k-2})\ldots(1+q)$$

$$= q^k(-q;q)_{k-1}$$

and summing over $k \in \mathbb{N}^*$ gives

$$\sum_{n\geq 0}p(\mathcal{D},n)q^n = 1 + \sum_{k>0}q^k(-q;q)_{k-1}$$

(so the sum on the left is equal to $(-q;q)_\infty$, the generating function for $p(\mathcal{D},n)$, see (1.13) above). Similarly,

$$\sum_{n\geq 0}(p(\mathcal{D}'_e,n) - p(\mathcal{D}'_o,n))q^n = 1 + \sum_{k>0}(-1)^k q^k(-q;q)_{k-1} \tag{2.6}$$

which, by (2.5) is equal to $1 - q + q^2 - q^5 + \ldots$.

2.2 Sylvester's bijection

The equality between the generating functions for the sequences $p(\mathcal{D}, n)$ and $p(\mathcal{O}, n)$ is identity (1.18). The formula for the difference between the number of partitions in $\mathcal{O}$ with first part congruent to 1 mod 4 and those with first part congruent to 3 mod 4, $p(\mathcal{O}_1, n) - p(\mathcal{O}_3, n)$, will be given later (3.8). But first a combinatorial proof that $p(\mathcal{D}, n) = p(\mathcal{O}, n)$ is presented.

Sylvester constructed a weight preserving bijection from the set of partitions into odd parts to the set of partitions into distinct parts. The bijection, $T : \mathcal{D} \to \mathcal{O}$, described here is the inverse of Sylvester's.

A good way to illustrate the nature of the map T is through an example of how it acts on the graph of a given partition. Let $\lambda = (19, 18, 17, 16, 15, 12, 9, 5, 2)$. Below is the graph of λ, or at least what would be the graph if the letters, which are used here because they make the description of T slightly clearer, were replaced by dots.

$$\begin{array}{lllllllllllllllllll}
a & a & w & w & w & b & b & b & b & x & x & x & c & c & c & y & d & z & e \\
a & a & w & w & w & b & b & b & b & x & x & x & c & c & c & y & d & z & \\
a & a & w & w & w & b & b & b & b & x & x & x & c & c & c & y & d & & \\
a & a & w & w & w & b & b & b & b & x & x & x & c & c & c & y & & & \\
a & a & w & w & w & b & b & b & b & x & x & x & c & c & c & & & & \\
a & a & w & w & w & b & b & b & b & x & x & x & & & & & & & \\
a & a & w & w & w & b & b & b & b & & & & & & & & & & \\
a & a & w & w & w & & & & & & & & & & & & & & \\
a & a. & & & & & & & & & & & & & & & & &
\end{array}$$

So instead of dots the columns of odd height are composed of letters from the start of the alphabet (each dot in the first such column is replaced by an a, each dot in the second such column is replaced by a b, and so on). The columns of even height consist of letters from the end of the alphabet, z for dots in the columns (or column in this case) of height one, y for each dot in the columns of height three and so on.

Now, the 'double block' of 9 as is transposed to give two columns of as, both of length 9. This can be written as $(2, 9|a) \to (9, 2|a)$. Likewise $(4, 7|b) \to (7, 4|b)$, $(3, 5|c) \to (5, 3|c)$, $(1, 3|d) \to (3, 1|d)$ and $(1, 1|e) \to (1, 1|e)$. The block of 24 ws is halved in height and doubled in length, $(3, 8|w) \to (6, 4|w)$, similarly $(3, 6|x) \to (6, 3|x)$, $(1, 4|y) \to (2, 2|y)$ and $(1, 2|z) \to (2, 1|z)$. The resulting blocks are stacked in such a way that the left most column has the e at the top, to its right are two columns, both with ds at the top and so on to the last pair of columns with as at the top (see below). Also the

ws, xs, ys and zs are aligned so to produce one row starting with the letter z at its left, the next row starts with y, the next row after this starts with x and the last one starts with w, as shown below,

$$
\begin{array}{cccccccccccccccccc}
e & & z & z & y & y & x & x & x & x & x & x & w & w & w & w & w & w \\
d & d & d & & y & y & x & x & x & x & x & x & w & w & w & w & w & w \\
c & c & c & c & c & & x & x & x & x & x & x & w & w & w & w & w & w \\
c & c & c & c & c & & & & & & & & w & w & w & w & w & w \\
c & c & c & c & c \\
b & b & b & b & b & b & b \\
b & b & b & b & b & b & b \\
b & b & b & b & b & b & b \\
b & b & b & b & b & a & b \\
a & a & a & a & a & a & a & a & a \\
a & a & a & a & a & a & a & a & a
\end{array}
.
$$

Now, raise the column with e at the top to the point $(1,-1)$ and slide the z column along, so it starts at $(1,-2)$. Then raise the next two columns so the two top ds occupy the points $(2,-2)$ and $(3,-2)$ and slide the next row in so the first y is at $(4,-2)$. Then raise the fourth and fifth columns so the two top cs in these colums are at $(4,-3)$ and $(5,-3)$ and slide the row starting with x along so that the first x is at $(6,-3)$. Then raise the next two columns so that the two top bs occupy $(6,-4)$ and $(7,-4)$ and slide the row of six ws along so the leftmost of these is at $(8,-4)$ and finally insert the remaining four as as shown,

$$
\begin{array}{ccccccccccccccccc}
e & z & z & y & y & x & x & x & x & x & x & w & w & w & w & w & w \\
d & d & d & y & y & x & x & x & x & x & x & w & w & w & w & w & w \\
c & c & c & c & c & x & x & x & x & x & x & w & w & w & w & w & w \\
c & c & c & c & c & b & b & w & w & w & w & w & w \\
c & c & c & c & c & b & b & a & a \\
b & b & b & b & b & b & b & a & a \\
b & b & b & b & b & b & b \\
b & b & b & b & b & a & a \\
b & b & b & b & b & a & a \\
a & a & a & a & a \\
a & a & a & a & a
\end{array}
.
$$

This diagram (viewed as a collection of dots, not letters) is the graph of $T(\lambda)$, so $T(\lambda) = (17,17,17,13,9,9,7,7,7,5,5)$.

The map T has some interesting properties. In the above example, the first part of $T(\lambda)$ is 17 and $T(\lambda)$ has 11 parts. Now $(17-1)/2+11 = 19$, which is λ_1, the first part of the original partition. This is true in general. It is also true that the number of sequences of consecutive integers in λ is the number of distinct odd numbers in $T(\lambda)$. This is an exercise in chapter 7 of [17], which also defines the bijection of Sylvester. An immediate consequence is that the number of partitions of n into odd parts (repetition allowed), where there are k distinct parts is the same as the number of partitions of n into distinct parts where the are k sequences of consecutive numbers, a proof based on generating functions is given in [2]. As an aside, it is worth noting that the partitions for which $T(\lambda) = \lambda$ are those of the form $(2n-1, 2n-3, ..., 3, 1)$.

There is another weight preserving bijection from $\mathcal{D}$ to $\mathcal{O}$. It is defined by breaking each even part, $n = a2^b$ of λ, into 2^b copies of a (where a is the largest odd divisor of n). Thus $(19,18,17,16,15,12,9,5,2) \rightarrow (19,9^2,17,1^{16},15,3^4,9,5,1^2) = (19,17,15,9^3,5,3^4,1^{18})$, where the powers denote multiplicities.

Chapter 3

A combinatorial approach to some partition identities

3.1 The three identities

The following identities are to be proved in this chapter. The identities are not new, but the proofs given here are.

$$\sum_{n\geq 0}(-1)^n\frac{q^n}{(-q^2;q^2)_n}=1-\sum_{m>0}q^{\frac{m(3m-1)}{2}}(1-q^m), \tag{3.1}$$

$$\sum_{n\geq 0}\frac{q^n}{(-q;q^2)_n}=1+q\sum_{m\geq 0}(-1)^m q^{2m(3m+2)}(1+q^{4m+2}), \tag{3.2}$$

$$\sum_{n\geq 0}q^n(q;q^2)_n=\sum_{m\geq 0}(-1)^m q^{m(3m+2)}(1+q^{2m+1}). \tag{3.3}$$

The proofs presented here involve considering partitions into odd parts and partitions into distinct parts.

Identity (3.1) is equivalent to identity (23.2) in [11], due to the fact that

$$1+\sum_{n>0}(-1)^{n-1}\frac{q^{2n-1}}{(q;q^2)_n}=\sum_{m\geq 0}\frac{q^m}{(-q^2;q^2)_m}.$$

This is so because each side of the above equation is the generating function for $p(\mathcal{O}_1,n)-p(\mathcal{O}_3,n)$ (see (3.8) below). Identity (3.2) is equivalent to (26.96) and (27.97) in [11].

The approach taken here is different from that of Fine. Whereas he uses the Hypergeometric series to prove (3.1) and (3.2) (and lots more besides), here all that is used are combinatorial arguments (adapting the Franklin bijection) and q-binomial coefficients.

3.2 A combinatorial proof of a result of Fine

There is, as indicated earlier, an expression similar to (1.16) but involving partitions into odd parts. It was found by Fine using hypergeometric series, see 23.7 in [11]. In this section a weight preserving involution, G, is constructed that acts on most of the partitions into odd parts. It also has the property that (on all the partitions on which it is defined) it reverses the residue mod 4 of the first part of the partition (this is the raison d'etre of the map G). Just as the Franklin bijection depends on whether the partition satisfies $\sigma(\lambda) < s(\lambda)$ or not, it is necessary to know if a certain inequality holds (for a given partition) in order to determine the action of G on that partition. In order to discuss the terms in this inequality it is necessary to introduce some new notation.

For λ a given partition (into odd parts), let t be such that $\lambda_t \geq 2t-1$ but $\lambda_{t+1} \leq 2t-1$, $e(\lambda) := \lambda_t - 2t + 1$ and define $d(\lambda)$ as the maximum l such that $\lambda_{t+l} = 2t-1$. Also for i from 1 to t let $x_\lambda(2i-1) := \#\{l : \lambda_l = 2i-1\}$ and $x_\lambda(2i) := \lambda_i - \lambda_{i+1}$. Now, define $r(\lambda)$, $\Sigma(\lambda)$ and $S(\lambda)$ by

$$r(\lambda) := \begin{cases} 2t-1 & \text{if } e(\lambda) = 0, \\ 2t & \text{if } e(\lambda) > 0, \end{cases}$$

$$\Sigma(\lambda) := \min(j : x_\lambda(j) > 0),$$

$$S(\lambda) := \begin{cases} d(\lambda)+1 & \text{if } e(\lambda) = 0, \\ e(\lambda)/2 & \text{if } e(\lambda) > 0. \end{cases}$$

In order to define G on a given partition $\lambda \in \mathcal{O}$ it is necessary to know if $\Sigma(\lambda) < S(\lambda)$ or $\Sigma(\lambda) \geq S(\lambda)$. But, whereas in the case of the Franklin map there were only two possibilities, for G there are eight possibilities. Any given partition, $\lambda \in \mathcal{O}$, is in precisely one of the following sets,

$$\begin{aligned}
A_1 &:= \{\lambda : \Sigma(\lambda) < S(\lambda), & \Sigma(\lambda) &= 2\sigma, & r(\lambda) &\equiv 0 \bmod 2\}, \\
A_2 &:= \{\lambda : \Sigma(\lambda) < S(\lambda), & \Sigma(\lambda) &= 2\sigma+1, & r(\lambda) &\equiv 0 \bmod 2\}, \\
A_3 &:= \{\lambda : \Sigma(\lambda) < S(\lambda), & \Sigma(\lambda) &= 2\sigma, & r(\lambda) &\equiv 1 \bmod 2\}, \\
A_4 &:= \{\lambda : \Sigma(\lambda) < S(\lambda), & \Sigma(\lambda) &= 2\sigma+1, & r(\lambda) &\equiv 1 \bmod 2\}, \\
B_1 &:= \{\lambda : \Sigma(\lambda) \geq S(\lambda), & S(\lambda) &= 2s, & r(\lambda) &\equiv 0 \bmod 2\}, \\
B_2 &:= \{\lambda : \Sigma(\lambda) \geq S(\lambda), & S(\lambda) &= 2s+1, & r(\lambda) &\equiv 0 \bmod 2\}, \\
B_3 &:= \{\lambda : \Sigma(\lambda) \geq S(\lambda), & S(\lambda) &= 2s, & r(\lambda) &\equiv 1 \bmod 2\}, \\
B_4 &:= \{\lambda : \Sigma(\lambda) \geq S(\lambda), & S(\lambda) &= 2s+1, & r(\lambda) &\equiv 1 \bmod 2\}.
\end{aligned}$$

Depending on whether $\lambda \in A_1, A_2, A_3, A_4, B_1, B_2, B_3$ or B_4, $G(\lambda)$ is defined (for most such partitions) as

$$(\lambda_1 - 4\sigma - 2, ..., \lambda_\sigma - 4\sigma - 2, \lambda_{\sigma+1} - 4\sigma, ..., \lambda_t - 4\sigma, \underbrace{2t+1, ..., 2t+1}_{2\sigma}, \lambda_{t+1}, ..., \lambda_k),$$

$$(\lambda_1 - 4\sigma - 2, ..., \lambda_t - 4\sigma - 2, \underbrace{2t+1, ..., 2t+1}_{2\sigma+1}, \lambda_{t+1}, ..., \lambda_{k-1}),$$

$$(\lambda_1 + 4\sigma - 2, ..., \lambda_\sigma + 4\sigma - 2, \lambda_{\sigma+1} + 4\sigma, ..., \lambda_t + 4\sigma, ..., \lambda_{t+2\sigma+1}, ..., \lambda_k),$$

$$(\lambda_1 + 4\sigma + 2, ..., \lambda_t + 4\sigma + 2, ..., \lambda_{t+2\sigma+2}, ..., \lambda_{k-1}),$$

$$(\lambda_1 - 4s + 2, ..., \lambda_s - 4s + 2, \lambda_{s+1} - 4s, ..., \lambda_t - 4s, \underbrace{2t-1, ..., 2t-1}_{2s}, \lambda_{t+1}, ..., \lambda_k),$$

$$(\lambda_1 - 4s - 2, ..., \lambda_t - 4s - 2, \underbrace{2t-1, ..., 2t-1}_{2s+1}, \lambda_{t+1}, ..., \lambda_{k-1}, \lambda_k, 2s+1),$$

$$(\lambda_1 + 4s + 2, ..., \lambda_s + 4s + 2, \lambda_{s+1} + 4s, ..., \lambda_{t-1} + 4s, ..., \lambda_{t+2s}, ..., \lambda_k),$$

$$(\lambda_1 + 4s + 2, ..., \lambda_{t-1} + 4s + 2, ..., \lambda_{t+2s+1}, ..., \lambda_{k-1}, \lambda_k, 2s+1).$$

The six dots ,......, are used to emphasise the omission of some entries, which can be seen by looking at the subscript. Writing $\lambda = (17, 17, 17, 13, 9, 9, 7, 7, 7, 5, 5)$, for example, gives $\lambda_5 = 9$, so $t = 5$. Now, $e(\lambda) = 0$ and $d(\lambda) = 1$. The sequence $x_\lambda(j)$ starts $(0, 0, 0, 0, 2, ...)$ and so has first nonzero entry at $j = 5$, giving $\Sigma(\lambda) = 5$. Since $e(\lambda) = 0$, it follows that $S(\lambda) = d(\lambda) + 1 = 2$ (and so $s = 1$). Hence $\Sigma(\lambda) \geq S(\lambda)$, $r(\lambda) = 9$ is odd but $S(\lambda)$ is even. This is case B_2, and $G((17, 17, 17, 13, 9, 9, 7, 7, 7, 5, 5)) = (17 + 6, 17 + 4, 17 + 4, 13 + 4, \hat{9}, \hat{9}, 7, 7, 7, 5, 5) = (23, 21, 21, 17, 7, 7, 7, 5, 5)$.

Clearly G reverses λ_1 mod 4. It is also a weight preserving involution. Thus, to find an expression for $p(\mathcal{O}_1, n) - p(\mathcal{O}_3, n)$ it is only necessary to consider the partitions for which G is not defined, these fall into four families, $\emptyset$, (7), (13^2), (19^3),....,$((6n+1)^n)$ and (1), (3^4), (5^7),....,$((2n+1)^{3n+1})$ and (1^2), (3^5), (5^8),....,$((2n+1)^{3n+2})$ and (5), (11^2), (17^3),....,$((6n+5)^{n+1})$.

Thus, if $\mathcal{O}_1 := \{\lambda \in \mathcal{O} : \lambda_1 \equiv 1 \bmod 4\}$ and $\mathcal{O}_3 := \{\lambda \in \mathcal{O} : \lambda_1 \equiv 3 \bmod 4\}$, then it follows that $p(\mathcal{O}_1, n) - p(\mathcal{O}_3, n) = 0$ unless $n = m(3m+1)/2$ whence,

$$p(\mathcal{O}_1, m(3m+1)/2) - p(\mathcal{O}_3, m(3m+1)/2) =$$

$$\begin{cases} 1 & \text{if } m = 0, 4, 8, ..., \text{ or } \pm 1, \pm 5, \pm 9, ..., \text{ or } -2, -6, -10, ... \\ -1 & \text{if } m = -4, -8, -12, ..., \text{ or } \pm 3, \pm 7, \pm 11, ..., \text{ or } 2, 6, 10, ... \end{cases} \tag{3.4}$$

Whereas the Franklin map is a fairly intuitive construction, as can be seen by considering the effect it has on the graph of the partition (this is done in all the references mentioned in section 2.1), the map G seems somewhat contrived. The idea behind G is simply that for $\lambda \in \mathcal{O}$, $G(\lambda) = TFT^{-1}(\lambda)$ (except for the exceptional partitions, for which the Franklin bijection is not defined). In fact, for $\lambda \in \mathcal{D}$, $\#\lambda = r(T(\lambda))$, $\sigma(\lambda) = \Sigma(T(\lambda))$ and $s(\lambda) = S(T(\lambda))$.

Finally note that (3.4) implies

$$\sum_{n \geq 0} (p(\mathcal{O}_1, n) - p(\mathcal{O})_3, n))q^n = 1 + q + q^2 + q^5 - q^7 - q^{12} - q^{15} - q^{22} + q^{26} + ... \tag{3.5}$$

3.3 A different approach

In this section a, rather long-winded, proof of (3.1) is given. The proof involves the concept of the hook, defined below, of a partition into odd parts.

The following identity will be needed, it follows from (21.21) in [11],

$$\sum_{i=0}^{n} q^i \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} = (-q; q)_n. \tag{3.6}$$

Now, recall that $\mathcal{U}(a, b)$ is the set of all partitions having no more than b parts, the largest of which is no greater than a. Similarly, let $\mathcal{U}' = \mathcal{U}'(2a, b)$ be the set of partitions into not more than b even parts, the largest of which is no greater than $2a$. It follows from (1.8) that

$$\sum_{\lambda \in \mathcal{U}'} q^{wt(\lambda)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_{q^2} \tag{3.7}$$

The next step is to obtain the generating function for $p(\mathcal{O}_1, n) - p(\mathcal{O}_3, n)$.

For $\lambda \in \mathcal{O}$, let $\epsilon_\lambda := (-1)^{(\lambda_1 - 1)/2}$. By considering the graph of all such λ it can be seen, where $\mathcal{O}(k)$ denotes the number of partitions into precisely k odd parts, that

$$\sum_{\lambda \in \mathcal{O}(k)} \epsilon_\lambda q^{wt(\lambda)} = q^k(1 - q^{2k} + q^{4k} + ...)(1 - q^{2(k-1)} + q^{4(k-1)} + ...)...(1 - q^2 + q^4 + ...)$$

$$= \frac{q^k}{(-q^2; q^2)_k}$$

hence (taking $\epsilon_\emptyset = 1$),

$$\sum_{k\geq 0}\sum_{\lambda\in\mathcal{O}(k)} \epsilon_\lambda q^{wt(\lambda)} = \sum_{k\geq 0}\frac{q^k}{(-q^2;q^2)_k}$$

and so,

$$\sum_{n\geq 0}(p(\mathcal{O}_1,n)-p(\mathcal{O}_3,n))q^n = \sum_{m\geq 0}\frac{q^m}{(-q^2;q^2)_m}. \tag{3.8}$$

As was seen in section 2.1.1., similar arguments give (2.6) which, to recap, states,

$$\sum_{n>0}(p(\mathcal{D}'_e,n)-p(\mathcal{D}'_o,n))q^n = \sum_{m>0}(-1)^m q^m(-q;q)_{m-1}. \tag{3.9}$$

Now the sum on the left of (3.8) is the sum over partitions in $\mathcal{O}$ of $(-1)^{\frac{\lambda_1-1}{2}}q^{wt(\lambda)}$. The number of parts of such a partition is congruent mod 2 to its weight. So $q \to -q$ in identity (3.8) gives

$$\sum_{\lambda\in\mathcal{O}}(-1)^{\frac{\lambda_1-1}{2}+\#(\lambda)}q^{wt(\lambda)} = \sum_{m\geq 0}(-1)^m\frac{q^m}{(-q^2;q^2)_m}. \tag{3.10}$$

Now, define $h(\lambda) := (\lambda_1-1)/2+\#(\lambda)$. This is a sort of modified hook (the hook is usually $\lambda_1 - 1 + \#(\lambda)$, but here $h(\lambda)$ is only defined for partitions into odd parts). A partition, λ, has $h(\lambda) = m$ if and only if both (A) $\lambda_1 = 2i+1$ and (B) $\#(\lambda) = m-i$, for some i for which $m > i \geq 0$. For a given i, summing over all partitions (into odd parts) which satisfy both (A) and (B) gives (using (3.7)),

$$\sum q^{wt(\lambda)} = q^{m+i}\begin{bmatrix} m-1 \\ i \end{bmatrix}_{q^2}$$

and summing over i gives from 0 to $m-1$ gives

$$\sum_{h(\lambda)=m} q^{wt(\lambda)} = \sum_{i=0}^{m-1} q^{m+i}\begin{bmatrix} m-1 \\ i \end{bmatrix}_{q^2}$$

and by (3.6) this becomes

$$\sum_{h(\lambda)=m} q^{wt(\lambda)} = q^m(-q;q)_{m-1}$$

considering the partitions with $h(\lambda) = 1, 2, \ldots$ gives the following identity

$$\sum_{\lambda\in\mathcal{O}^*} z^{h(\lambda)}q^{wt(\lambda)} = \sum_{m>0} z^m q^m(-q;q)_{m-1}$$

where $\mathcal{O}^* := \mathcal{O}\setminus\emptyset$. Setting $z = -1$ and taking $h(\emptyset) = 1$ gives

$$\sum_{\lambda\in\mathcal{O}}(-1)^{h(\lambda)}q^{wt(\lambda)} = 1 + \sum_{m>0}(-1)^m q^m(-q;q)_{m-1}. \tag{3.11}$$

The sum on the right of (3.11) is that in (3.9). As was noted at the end of chapter 2, the sum on the left of (2.6) (and so (3.9)) is $1 - q + q^2 - q^5 + \ldots$. So, given (3.10), (3.1) is proved. Note that, having done all this, (3.1) follows directly from $q \to -q$ in (3.5) and (3.8) (so actually there is no need to invoke (3.6) or the hook of a partition).

3.4 The second and third identities

In this section identities (3.2) and (3.3) are proved. The method of proof requires introducing the (new) concept of nested partitions. Then it is shown that the generating function for nested partitions with those into an odd number of parts counted negatively is the sum on the left hand side of (3.2). Next, using two elementary identities ((3.12) and (3.16) below), it is shown that this simplifies to $\sum q^{2m+1}(q^2;q^4)_m$ which is essentially the sum on the left in (3.3). This being so means that if (3.3) were to be proved, then so would (3.2). The proof of (3.3) relies on the fact that the sum on the left of this identity is also related to nested partitions: it is the generating function for nested partitions into distinct parts, again those into an odd number of parts being counted negatively. Finally an involution is defined for most such nested partitions, proving (3.3) (and so also proving (3.2)).

Having outlined the method of this section it is now necessary to introduce the two q-binomial elementary identities required here, this is done in the next subsection. Note that the two identities form part of theorem 3.4 in [2] where they are proved, and attributed to Gauss (the proof given here of (3.12) is different to that in [2]).

3.4.1 Two q-binomial identities

The first identity is

$$\sum_{i=0}^{m} q^i \begin{bmatrix} n+i \\ i \end{bmatrix} = \begin{bmatrix} n+m+1 \\ m \end{bmatrix}. \tag{3.12}$$

This is a consequence of the lemma in section 1.2.2. It follows from that lemma (identity (1.8)) that the generating function for partitions with no more than $n+1$ parts and first part no greater than m is $\begin{bmatrix} n+m+1 \\ m \end{bmatrix}$, which is to say that

$$\sum_{\lambda\in\mathcal{U}(m,n+1)} q^{wt(\lambda)} = \begin{bmatrix} n+m+1 \\ m \end{bmatrix}. \tag{3.13}$$

Now, if $\mathcal{U}_i(m,n+1)$ is defined to be the set of such partitions having first part $\lambda_1 = i$ then it is clear that $\mathcal{U}(m,n+1)$ is the disjoint union of the sets $\mathcal{U}_i(m,n+1)$ (as i ranges from 0 to m). So

$$\sum_{\lambda\in\mathcal{U}(m,n+1)} q^{wt(\lambda)} = \sum_{i=0}^{m}\left(\sum_{\lambda\in\mathcal{U}_i(m,n+1)} q^{wt(\lambda)}\right) \tag{3.14}$$

and a partition in $\mathcal{U}_i(m,n+1)$ has (by definition) a first part of size i and no more than n additional parts. Hence

$$\sum_{\lambda\in\mathcal{U}_i(m,n+1)} q^{wt(\lambda)} = q^i \sum_{\lambda\in\mathcal{U}(i,n)} q^{wt(\lambda)}, \tag{3.15}$$

the sum on the right in the above equation is clearly equal to $\left[{n+i \atop i}\right]$ (by (1.8), again). Thus (3.15) becomes

$$\sum_{\lambda \in \mathcal{U}_i(m,n+1)} q^{wt(\lambda)} = q^i \begin{bmatrix} n+i \\ i \end{bmatrix}$$

which, together with (3.14), implies that

$$\sum_{\lambda \in \mathcal{U}(m,n+1)} q^{wt(\lambda)} = \sum_{i=0}^{m} q^i \begin{bmatrix} n+i \\ i \end{bmatrix}.$$

Finally, the fact that the sum on the right in the above equation is (by (1.8), once again) equal to $\left[{n+m+1 \atop m}\right]$ implies the veracity of identity (3.12), as desired.

The second identity is

$$\sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} = \begin{cases} (q;q^2)_m & \text{if } n = 2m, \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.16}$$

This, like the previous identity, is proved in [2] where it is part of theorem 3.4.

3.4.2 Nested partitions

A nested partition is here defined as a sequence, $\lambda = (\lambda_0, [\lambda_1], ..., [\lambda_k])$ where (for i such that $0 \leq i < k$) $\lambda_i \geq \lambda_{i+1} > 0$. For $i > 0$, the entry $[\lambda_i]$ signifies the pair $(\lambda_i, \lambda_i - 1)$, and will be called a nested entry (so the only non-nested entry is λ_0). The weight of λ is defined as $wt(\lambda) = \lambda_0 - k + 2\sum_{i=1}^{k} \lambda_i$ where k, clearly, is the number of (nested) parts. A typical nested partition is $\lambda = (9, [9], [9], [6], [4], [4], [1], [1], [1])$. Now $k = 8$ but there are 14 rows in the graph, each nested entry greater than 1 is represented by two rows, the second entry in the second row has an empty dot in its place, thus the graph for λ defined above is

$$\begin{array}{ccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \circ & \bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & \bullet & & & & & \\
\bullet & \circ & \bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & \bullet & & & & & \\
\bullet & \circ & \bullet & \bullet & & & & & \\
\bullet & & & & & & & & \\
\bullet & & & & & & & & \\
\bullet & & & & & & & &
\end{array}$$

so $wt(\lambda) = 71$ (the number of black dots). Let A denote the set of all (nonempty) nested partitions. For $\lambda \in A$, define $\hbar(\lambda) := \lambda_0 + k$ (this is the hook for nested partitions). Now $\hbar(\lambda) = a + b + c$ where $a = \lambda_0$, $b = \#\{i > 0 : \lambda_i > 1\}$ and $c = \#\{j > 0 : \lambda_j = 1\}$. Such a partition is said to have type (a, b, c), for instance, in the above example λ has type $(9, 5, 3)$. Considering all nested partitions of type $(9, 5, 3)$

$$\sum_{\lambda \to (9,5,3)} q^{wt(\lambda)} = q^{27} \begin{bmatrix} 12 \\ 5 \end{bmatrix}_{q^2}.$$

To see this, consider the graph for $(9, [2], [2], [2], [2], [2], [1], [1], [1])$, which has weight 27. There is a 7 by 5 rectangle that can be filled (by pairs of dots) in various ways. Every nested partition of type $(9, 5, 3)$ arises this way. This is basically the same idea as that in section 3.3 where it was observed that for a partition in $\mathcal{O}$ to have a given value for its hook, it must satisfy two conditions. Now

$$\frac{q^n}{(-q; q^2)_n} = q^n(1 - q^{2n-1} + q^{4n-2} + ...)(1 - q^{2n-3} + q^{4n-6} + ...)...(1 - q + q^2 + ...)$$

which implies

$$\sum_{n>0} \frac{q^n}{(-q; q^2)_n} = \sum_{\lambda \in A} (-1)^k q^{wt(\lambda)}$$

So the approach is: for a given l, investigate all partitions with $\hbar(\lambda) = l$ and sum over l. Suppose $l = a + b + c$. Then there are a dots in the top row (of the graph), $3b$ (nonempty) dots for the b nested parts that are > 1 (because there are three dots for each such part) and c dots for the c parts that are equal to 1. In the above example $a + 3b + c = 27$. Also there is, in general, a $(a - 2)$ by b rectangle that may be filled with pairs of dots. So, counting those partitions where k is odd negatively gives

$$\sum_{\lambda \to (a,b,c)} (-1)^k q^{wt(\lambda)} = (-1)^{b+c} q^{a+3b+c} \begin{bmatrix} a + b - 2 \\ b \end{bmatrix}_{q^2}$$

for a fixed, summing over b gives

$$\sum_{b=0}^{l-a} \Big(\sum_{\lambda \to (a,b,c)} (-1)^k q^{wt(\lambda)} \Big) = (-1)^{l-a} q^l \sum_{b=0}^{l-a} q^{2b} \begin{bmatrix} a + b - 2 \\ b \end{bmatrix}_{q^2}.$$

By (3.12) this can be written as

$$\sum_{b=0}^{l-a} \Big(\sum_{\lambda \to (a,b,c)} (-1)^k q^{wt(\lambda)} \Big) = (-1)^{l-a} q^l \begin{bmatrix} l - 1 \\ l - a \end{bmatrix}_{q^2}$$

and summing over a gives

$$\sum_{a=1}^{l} \Big(\sum_{b=0}^{l-a} \Big(\sum_{\lambda \to (a,b,c)} (-1)^k q^{wt(\lambda)} \Big) \Big) = (-1)^l q^l \sum_{a=1}^{l} (-1)^a \begin{bmatrix} l - 1 \\ l - a \end{bmatrix}_{q^2}.$$

Now, by (3.16) the expression on the right is $q^{2m+1}(q^2;q^4)_m$ when $l = 2m+1$ and 0 when l is even. Thus, for $l = 2m+1$

$$\sum_{\hbar(\lambda)=l} (-1)^k q^{wt(\lambda)} = q^{2m+1}(q^2;q^4)_m$$

so summing over l gives

$$\sum_{\lambda \in A} (-1)^k q^{wt(\lambda)} = \sum_{m>0} q^{2m+1}(q^2;q^4)_m.$$

Thus identity (3.2) has been shown to follow from identity (3.3) (by $q \to q^2$ and then multiplication by q).

In order to prove (3.3), it is necessary to consider nested partitions into distinct parts. Let B be the set of these partitions, that is $B = \{\lambda \in A | i > 0 \Rightarrow \lambda_i > \lambda_{i+1}\} \cup \{\emptyset\}$. Note that $\lambda_0 = \lambda_1$ is allowed. Now,

$$\begin{aligned} q^n(q;q^2)_n &= q^n(1 - q^{(n)+(n-1)})(1 - q^{(n-1)+(n-2)})...(1 - q^{(1)+(0)}) \\ &= \sum_{\lambda \in B(n)} (-1)^k q^{wt(\lambda)} \end{aligned}$$

where $B(n) := \{\lambda \in B : \lambda_0 = n\}$ (with $B(0) = \{\emptyset\}$). Summing over n gives

$$\sum_{n \geq 0} q^n(q;q^2)_n = \sum_{\lambda \in B} (-1)^k q^{wt(\lambda)}. \tag{3.17}$$

This means that proving

$$\sum_{\lambda \in B} (-1)^k q^{wt(\lambda)} = \sum (-1)^m q^{m(3m+2)}(1 + q^{2m+1}) \tag{3.18}$$

is equivalent to proving (3.3) (and so (3.2)). In order to prove (3.18) it is necessary to construct a map $\lambda \to \lambda'$. This map plays the same role as $\lambda \to F(\lambda)$ (Franklins' bijection) defined on normal (i.e. not nested) partitions into distinct parts in chapter 2 and $\lambda \to G(\lambda)$ defined on (normal) partitions into odd parts earlier in this chapter. In particular, $wt(\lambda') = wt(\lambda)$, $\sigma(\lambda) < s(\lambda) \Leftrightarrow \sigma(\lambda') \geq s(\lambda')$ (σ and s for nested partions are defined below), $\lambda \to \lambda' \to \lambda'' = \lambda$, and λ' is defined for most such partitons.

The proof is similar to Franklin's. Define

$$s = s(\lambda) := \begin{cases} 2\lambda_k - 1 & \text{if } k > 0 \\ \infty & \text{if } k = 0 \end{cases}$$

and, if i is defined to be 0 if $\lambda_0 \neq \lambda_1$ and as the maximum j such that $\lambda_0 - \lambda_j = j - 1$ if $\lambda_0 = \lambda_1$ then

$$\sigma = \sigma(\lambda) := 2i + 1.$$

Now, define λ' by

$$\lambda' := \begin{cases} (\lambda_0 - 1, [\lambda_1 - 1], ..., [\lambda_i - 1], [\lambda_{i+1}], ..., [\lambda_k], [i+1]) & \text{if } \sigma < s, \\ (\lambda_0 + 1, [\lambda_1 + 1], ..., [\lambda_{\lambda_k - 1} + 1], [\lambda_{\lambda_k}], ..., [\lambda_{k-1}], \hat{\lambda_k}) & \text{if } \sigma \geq s. \end{cases}$$

Note that if $\lambda = (n)$ (which happens if $k = 0$) and $n > 1$ then $\lambda' = (n, [1])$. The only other partition where $k < 1$ and λ' is undefined is the empty partition.

All that needs to be done is identify the exceptional partitions (for which $k > 0$). Now if $\sigma(\lambda) < s(\lambda)$ then $2i+1 < 2\lambda_k - 1$ so $i+1 < \lambda_k$. The only possible problem (in the case $\sigma < s$) is at $i+1 = \lambda_k - 1$. For the penultimate entry in λ' is either $[\lambda_k]$ or $[\lambda_k - 1]$ and the last is $[i+1]$, which is not a problem if $i+2 < \lambda_k$. However if $i+2 = \lambda_k$, then the last entries in λ' could be $[\lambda_k - 1]$ followed by $[i+1]$. This would occur if and only if $i = k$, which is equivalent to saying that $\lambda_p = \lambda_{p+1} + 1$ for all $p = 1, 2, ..., k-1$ (and that $\lambda_0 = \lambda_1$). This reduces the set of possible exceptional (for the case $\sigma < s$) partitions to those of the form $(n, [n], [n-1], ..., [m])$. In short, what is required is to find all such partitons satisfying both (a) $\sigma < s$ and (b) $i+1 = \lambda_k - 1$. But (for such partitions), $i = n-m+1$, $\lambda_k = m$ so $s = 2m-1$ and $\sigma = 2n-2m+3$. Hence (a) gives $n < 2m-2$ and, more importantly, (b) becomes $n = 2m-3$. Clearly, for $\sigma < s$, the only problematic partitions are $(3, [3])$, $(5, [5], [4])$,...,$(2m-3, [2m-3], ..., [m])$. These partitions have $k = m-2$ and $wt(\lambda) = (m-1)(3m-5)$ (for $m > 2$).

Now, if $\sigma(\lambda) \geq s(\lambda)$ then $2\lambda_k - 1 \leq 2i+1$ so $\lambda_k - 1 \leq i$. Clearly $i \leq k$ with equality at (and only at) those partitions mentioned above that have successive entries differing by precisely 1 (except $\lambda_0 = \lambda_1$). So $\lambda_k - 1 \leq k$ and (from the definition of λ'), problems occur precisely when equality holds. This happens when $m-1 = n-m+1$ and so (for $s \leq \sigma$) the exceptional partitions are those of the form $(2, [2])$, $(4, [4], [3])$,...,$(2m-2, [2m-2], ...[m])$. These partitions have $k = m-1$ and $wt(\lambda) = (m-1)(3m-1)$ (for $m > 1$).

Let B' be the set of non exceptional partitions. Now, $\lambda \rightarrow \lambda'$ is indeed a weight preserving involution on B' with $\sigma(\lambda) < s(\lambda) \nLeftrightarrow s(\lambda) \leq \sigma(\lambda)$ and so

$$\sum_{\lambda \in B} (-1)^k q^{wt(\lambda)} = \sum_{\lambda \in B \backslash B'} (-1)^k q^{wt(\lambda)} + \sum_{\lambda \in B'} (-1)^k q^{wt(\lambda)}$$

$$= \sum_{\lambda \in B'} (-1)^k q^{wt(\lambda)} = 1 + q + \sum_{m>2} (-1)^{m-2} q^{(m-1)(3m-5)}$$

$$+ \sum_{m>1} (-1)^{m-1} q^{(m-1)(3m-1)}.$$

This proves (3.3), and so (3.2) is proved too.

3.5 Two identities, one old and one new

3.5.1 An identity of Subbaro and Vidyasagar

In [30] Subbaro and Vidyasagar prove the following identity:

$$\sum_{n\geq 0}(-1)^n\frac{z^{2n}q^{n^2+n}}{(zq;q^2)_{n+1}}=\sum_{m\geq 0}(-1)^m z^{3m}q^{3m^2+2m}(1+zq^{2m+1}) \tag{3.19}$$

(it is (1.5) in their paper, they use a and x instead of z and q). Their proof involves algebraic manipulation of power series such as these and relies on the triple product identity, and (a version of) what is known as the quintuple product identity. What follows here is a direct, combinatorial, proof of the above identity.

Firstly, for $n \in \mathbb{N}$ consider the set $S(n)$ defined by

$$S(n) := \{\mu = (\pi_0, \pi_1, \pi_1, \pi_2, \pi_2, ..., \pi_n, \pi_n) : \pi_0 \geq \pi_1 \geq ... \geq \pi_n \geq 0\}$$

(the reason why an arbitrary element of this set is denoted by μ, whilst its parts are labelled π_i will become clear shortly). The elements of $S(n)$ can be viewed as partitions having the property the second part is equal to the third, the fourth part equal to the fifth, and so on. Viewed in this light it is seen that the dual of an element of $S(n)$ is a partition into odd parts, the biggest of which does not exceed $2n+1$ (the largest part of the dual of an element $\mu \in S(n)$ is $2n+1$ if, and only if, $\pi_n \neq 0$). It follows that

$$\frac{1}{(q;q^2)_{n+1}}=\sum_{\mu\in S(n)} q^{wt(\mu)}$$

(where the weight is the sum of the parts of μ, $wt(\mu) = \pi_0 + 2\pi_1 + 2\pi_2 + ... + 2\pi_n$). In fact, since the first part (i.e. π_0) of an element of $S(n)$ is the number of parts of the element's dual,

$$\frac{1}{(zq;q^2)_{n+1}}=\sum_{\mu\in S(n)} z^{\pi_0}q^{wt(\mu)}$$

hence

$$\frac{z^{2n}q^{n^2+n}}{(zq;q^2)_{n+1}}=\sum_{\mu\in S(n)} z^{\pi_0+2n}q^{wt(\mu)+n^2+n}.$$

which implies that

$$\sum_{n\geq 0}(-1)^n\frac{z^{2n}q^{n^2+n}}{(zq;q^2)_{n+1}}=\sum_{n\geq 0}(-1)^n\left(\sum_{\pi\in S(n)} z^{\pi_0+2n}q^{wt(\pi)+n^2+n}\right). \tag{3.20}$$

Now if $T(n)$ is defined to be the set of all ordered pairs with first part an element of $S(n)$ and second part the sequence $(2n, 2n-2, ..., 4, 2)$,

$$T(n) := \{\pi = (\mu, (2n, 2n-2, ..., 4, 2)) : \mu \in S(n)\}$$

(so $T(0)$ consists of all pairs $((\pi_0), \emptyset)$, $T(1)$ consists of all pairs of the form $((\pi_0, \pi_1, \pi_1), (2))$ etc) and, for $\pi = ((\pi_0, \pi_1, \pi_1, ..., \pi_n, \pi_n), (2n, 2n-2, ..., 4, 2)) \in T(n)$, the statistics $k(\pi)$ and $l(\pi)$ are defined as

$$k(\pi) := n, \qquad l(\pi) := \pi_0 + 2n$$

and the Weight, $Wt(\pi)$ as

$$Wt(\pi) := wt(\pi) + n^2 + n.$$

then it follows that

$$(-1)^n \sum_{\pi \in S(n)} z^{\pi_0 + 2n} q^{wt(\pi)+n^2+n} = \sum_{\pi \in T(n)} (-1)^{k(\pi)} z^{l(\pi)} q^{Wt(\pi)}. \tag{3.21}$$

Hence, defining T to be the union of all $T(n)$,

$$T := \{\pi = (\pi_0, \pi_1, \pi_1, ..., \pi_n, \pi_n), (2n, 2n-2, ..., 4, 2) : n \geq 0\}$$

it follows that, after summing over n in (3.21), (3.20) becomes

$$\sum_{\pi \in T} (-1)^{k(\pi)} z^{l(\pi)} q^{Wt(\pi)} = \sum_{n \geq 0} (-1)^n \frac{z^{2n} q^{n^2+n}}{(zq; q^2)_{n+1}}.$$

So, in order to prove (3.19), it suffices to show that

$$\sum_{\pi \in T} (-1)^{k(\pi)} z^{l(\pi)} q^{Wt(\pi)} = \sum_{m \geq 0} (-1)^m z^{3m} q^{3m^2+2m} (1 + zq^{2m+1}) \tag{3.22}$$

Proof of (3.22): An involution $\pi \rightarrow \pi'$ is to be defined on (most of) T. The involution will be Weight preserving, i.e. $Wt(\pi') = Wt(\pi)$. The involution will also have the property that $l(\pi') = l(\pi)$, but $k(\pi') \not\equiv k(\pi) \bmod 2$ (ensuring that $(-1)^{k(\pi')} \neq (-1)^{k(\pi)}$). As usual there will be a small subset $X \subset T$ containing the exceptional elements on which it is impossible to define the involution, and the sum on the left of (3.22) reduces to a sum over X (on account of most elements π being paired off with a corresponding π').

As usual the involution depends on whether or not an inequality of the form $\sigma(\pi) < s(\pi)$ is satisfied. In this case the two statistics are given, for $\pi = ((\pi_0, \pi_1, \pi_1, ..., \pi_n, \pi_n), (2n, 2n-2, ..., 2))$, by

$$\sigma(\pi) := \max(r : \pi_r = \pi_0) \quad \text{and} \quad s(\pi) := \pi_n$$

and define the sets A an B as

$$A := \{\pi \in T : \sigma(\pi) < s(\pi)\} \quad \text{and} \quad B := \{\pi \in T : \sigma(\pi) \geq s(\pi)\}.$$

A bijection between these two sets, or at least bijection that works for all elements except those that give rise to the expression on the right side of (3.22)) is required. This would be an involution on T. The proposed involution, $\lambda \rightarrow \lambda'$, is defined as

$$\pi' :=$$
$$\begin{cases} ((\pi_0-2,\pi_1-2,\pi_1-2,...,\pi_\sigma-2,\pi_\sigma-2,\pi_{\sigma+1}-1,\pi_{\sigma+1}-1,...,\pi_n-1,\pi_n-1,\sigma,\sigma), \\ (2n+2,2n,...,2)) & \text{if } \pi \in A \\ ((\pi_0+2,\pi_1+2,\pi_1+2,...,\pi_s+2,\pi_s+2,\pi_{s+1}+1,\pi_{s+1}+1,...,\pi_{n-1}+1,\pi_{n-1}+1), \\ (2n-2,2n-4,...,2)) & \text{if } \pi \in B \end{cases}$$

It is now necessary to establish for which $\pi \in T$ the map $\pi \to \pi'$ is not defined on:

Firstly, for $\pi \in B$ it is clear that if $\sigma(\pi) < n$ then the map $\pi \to \pi'$ is defined (by definition the slope $\sigma(\pi)$ cannot be greater than the number of parts, n). If π is such that $\sigma(\pi) = n$ then $\pi_n \leq n$, for if $\pi_n = m > n$ (and $\sigma(\pi) = n$) then $\pi = ((m,m,m,...m,m),(2n,2n-2,...4,2))$ and then $\sigma(\pi) = n < m = s(\pi)$ and so $\pi \in A$. For π such that $\sigma(\pi) = n$, $\pi \to \pi'$ is defined if and only only if $\pi_n < n$. Thus the elements $((0),\emptyset),((1,1,1),(2)),((2,2,2,2,2),(4,2)),...$ are those in B for which the map is undefined. These form X_0 where

$$X_0 = \{((n,n,n,...,n,n),(2n,2n-2,...,4,2))\}.$$

Secondly if $\pi \in A$ then the definition of σ implies that $\pi_\sigma > \pi_{\sigma+1}$ and so $\pi_\sigma - 2 > \pi_{\sigma+1} - 1$, which is required (at least if $\sigma(\pi) < n$). Furthermore, for $\pi \in A$, $s(\pi) > \sigma(\pi)$ and so $\pi_n - 1 \geq \sigma$, so there is no problem unless $\sigma(\pi) = n$. If $\pi \in A$ is such that $\sigma(\pi) = n$ then the first half of the sequence π' ends $(...,\pi_n - 2,\sigma,\sigma) = (...,\pi_n-2,n,n)$. Thus, for $\pi \in A$, there is a problem if $\sigma(\pi) = n$ and $\pi_n < n+2$. So suppose that π is such that $\sigma(\pi) = n$ and $\pi_n < n+1$, in this case $\sigma(\pi) = n \geq \pi_n = s(\pi)$ so $\sigma(\pi) \geq s(\pi)$ and $\pi \in B$. If, on the other hand, $\pi_n = n+1$, so $s(\pi) = n+1$ and $\sigma(\pi) = n$ then $s(\pi) > \sigma(\pi)$ and $\pi \in A$, and it is impossible to define the map $\pi \to \pi'$. These are the elements $((1),\emptyset),((2,2,2),(2)),((3,3,3,3,3),(4,2)),....$ Defining

$$X_1 := \{((n+1,n+1,n+1,...,n+1,n+1),(2n,2n-2,...,4,2))\},$$

it is clear these are the elements of A for which there is a problem. If X is the set of elements π for which the map is undefined then $X = X_0 \cup X_1$.

Now

$$\sum_{\pi\in T}(-1)^{k(\pi)}z^{l(\pi)}q^{Wt(\pi)} = \sum_{\pi\in T\setminus X}(-1)^{k(\pi)}z^{l(\pi)}q^{Wt(\pi)} + \sum_{\pi\in X}(-1)^{k(\pi)}z^{l(\pi)}q^{Wt(\pi)}$$

and the sum over $T \setminus X$ is 0, so

$$\sum_{\pi\in T}(-1)^{k(\pi)}z^{l(\pi)}q^{Wt(\pi)} = \sum_{\pi\in X}(-1)^{k(\pi)}z^{l(\pi)}q^{Wt(\pi)}$$

which is equal to

$$\sum_{\pi\in X_0}(-1)^{k(\pi)}z^{l(\pi)}q^{Wt(\pi)} + \sum_{\pi\in X_1}(-1)^{k(\pi)}z^{l(\pi)}q^{Wt(\pi)}$$

and the first sum is equal to $1 - z^3q^5 + z^6q^{16} + ...$ and the second to $zq - z^4q^8 + z^7q^{21} + ...$ This proves (3.19).

3.5.2 A new identity

The following identity is, I believe, new:

$$\sum_{n\geq 0}(-1)^n \frac{z^n q^{n^2}}{(zq;q^2)_{n+1}} = 1 \tag{3.23}$$

It is proved in a similar way to (3.19) above:

Let $\mu = (\pi_0, \pi_1, \pi_1, ..., \pi_n, \pi_n)$ as above. The set U is now defined to be

$$U = \{\pi = (\mu, (2n-1, 2n-3, ..., 3, 1)) : \mu \in S(n), n \in \mathbb{N}\}.$$

For example, $((0), \emptyset)$, $((4), \emptyset)$ and $((5,5,5,3,3),(3,1))$ are all elements of U.

The statistics $K(\pi)$ and $L(\pi)$ are defined, for $\pi \in U$, as

$$K(\pi) = n, \qquad L(\pi) = \pi_0 + n.$$

The weight, $W(\pi)$ is defined to be

$$W(\pi) = \pi_0 + 2\pi_1 + 2\pi_2 + ... + 2\pi_n + n^2.$$

Thus (3.23) is equivalent to

$$\sum_{\pi\in U}(-1)^{K(\pi)} z^{L(\pi)} q^{W(\pi)} = 1. \tag{3.24}$$

What is required is a weight preserving involution on U, $\pi \to \pi'$, for which $K(\pi') = K(\pi)$ and $L(\pi') \not\equiv L(\pi) \bmod 2$. The only element of U on which the map is undefined should be $\pi = ((0), \emptyset)$.

With this in mind the sets A and B are defined by

$$A := \{\pi \in U : \pi_n = 0\} \quad \text{and} \quad B := \{\pi \in U : \pi_n \neq 0\}.$$

Once this has been done, the map $\pi \to \pi'$ can be defined (the trick here is to notice that the first entry of the second part of any element π is $2n-1$ and the length of the first part is $2n+1$).

So, for $\pi = ((\pi_0, \pi_1, \pi_1, ..., \pi_n, \pi_n), (2n-1, 2n-3, ..., 3, 1)) \in U$, the map is defined as $\pi \to \pi'$ where π' is given by

$$\begin{cases} ((\pi_0+1, \pi_1+1, \pi_1+1, ..., \pi_{n-1}+1, \pi_{n-1}+1), (2n-3, 2n-5, ..., 3, 1)) \text{ if } \pi \in A, \\ ((\pi_0-1, \pi_1-1, \pi_1-1, ..., \pi_n-1, \pi_n-1, 0, 0), (2n+1, 2n-1, ..., 3, 1)) \text{ if } \pi \in B. \end{cases}$$

For instance, if $\pi = ((7,6,6,6,6,0,0),(5,3,1))$ then $\pi \in A$ and $\pi' = ((8,7,7,7,7),(3,1))$. It is clear that the only element for which the map is undefined is $\pi = ((0), \emptyset)$. It is also clear that the map has all the desired properties and so (3.24) is proved, and so therefore is (3.23).

Chapter 4

An involutive proof of the triple product identity

The triple product identity states that

$$[-z;q](q;q)_\infty = \sum_{n\in\mathbb{Z}} z^n q^{\frac{n^2-n}{2}} \tag{4.1}$$

(this is (1.17), but with z replaced by $-z$).

Now, the set $\mathcal{D}$ has already been defined as the set of strictly decreasing sequences of positive integers, i.e. $\mathcal{D}$ is the set of partitions into distinct parts. The set $\mathcal{D}_0$ is now defined to be the set of decreasing sequences of nonnegative integers (the set $\mathcal{D}_0$ is not to be confused with $\mathcal{D}_o$, the set of partitions into an odd number of distinct parts, which does not appear in this chapter). Clearly $\mathcal{D} \in \mathcal{D}_0$. The weight of a sequence in $\mathcal{D}_0$ is defined to be the sum of the parts of the sequence and for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in \mathcal{D}_0$, $\#(\alpha) := k$, the number of parts (these defintions are the same as for sequences in $\mathcal{D}$).

It follows that

$$\sum_{\alpha\in\mathcal{D}_0} z^{\#(\alpha)} q^{wt(\alpha)} = [-z;q]$$

and that

$$\sum_{\beta\in\mathcal{D}} z^{-\#(\beta)} q^{wt(\beta)} = [-z^{-1}q;q].$$

This means that if $J := \mathcal{D}_0 \times \mathcal{D} \times \mathcal{D}$, and for $\pi = (\alpha, \beta, \gamma) \in J$ (where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$, $\beta = (\beta_1, \beta_2, ..., \beta_l)$ and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_m)$), then by setting $c(\pi) := \#(\gamma) = m$ and $d(\pi) := \#(\alpha) - \#(\beta) = k - l$ allows the triple product identity (4.1), to be stated as

$$\sum_{\pi \in J} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)} = \sum_{n \in \mathbb{Z}} z^n q^{(n^2-n)/2} \tag{4.2}$$

where the weight of π is defined as the sum of the weights of its three parts: $wt(\pi) := wt(\alpha) + wt(\beta) + wt(\gamma)$.

A crucial step in this proof of the triple product identity, as in the case of Franklin's proof of the pentagonal number theorem, is finding (for any given $\pi \in J$) two statistics $s(\pi)$ and $\sigma(\pi)$ and a map $\pi \to \pi'$ (which is a weight preserving involution) such that the inequality $s(\pi) > \sigma(\pi)$ is satisfied precisely when $s(\pi') \leq \sigma(\pi')$.

It will be seen shortly that whether or not the inequality is satisfied is, in some cases, dependent on whether or not $\beta_1 > \gamma_1 + l - k$ (because this is $s(\pi) > \sigma(\pi)$). It will also be seen that, in some cases, it will be necessary to know whether or not $\alpha_k = 0$ and whether or not $\beta_l = 1$. Because of this it is necessary to adopt the following two conventions: Firstly, the first part of the empty sequence $\emptyset$ is 0 (it is perhaps better to say that the weight of the first part of $\emptyset$ is 0). For example, if $\beta = \emptyset$ (and γ is some other sequence) then $\beta_1 > \gamma_1 + l - k$ is equivalent to $k > \gamma_1$, because $\beta = \emptyset \Rightarrow \beta_1 = 0$ and $l = 0$, β_1 being the first part of β and l its length. Secondly, that the last part of an empty sequence is neither 0 or 1, i.e $\alpha = \emptyset \Rightarrow \alpha_k \neq 0$ or 1.

Now, for $\pi = ((\alpha_1, \alpha_2, ..., \alpha_k), (\beta_1, \beta_2, ..., \beta_l), (\gamma_1, \gamma_2, ..., \gamma_m)) \in J$, the two statistics are defined by

$$s(\pi) := \begin{cases} \infty & \text{if } \beta_1 > \gamma_1 + l - k,\ \alpha_k = 0,\ \beta = \emptyset,\ \gamma = \emptyset, \\ \gamma_m & \text{if } \beta_1 > \gamma_1 + l - k,\ \alpha_k = 0,\ \beta = \emptyset,\ \gamma \neq \emptyset, \\ \beta_1 & \text{else} \end{cases}$$

and

$$\sigma(\pi) := \begin{cases} \max(r : \alpha_1 - \alpha_r = r - 1) & \text{if } \beta_1 > \gamma_1 + l - k,\ \alpha_k = 0,\ \beta = \emptyset, \\ \gamma_1 + l - k & \text{else.} \end{cases}$$

Thus, if the set T is defined as $T := \{\pi : \beta_1 > \gamma_1 + l - k, \alpha_k = 0, \beta = \emptyset\}$ then $\pi \in T$ implies that $s(\pi)$ is either ∞ or the last entry in γ, depending on whether or not $\gamma = \emptyset$, and that $\sigma(\pi)$ is the maximum r such that $\alpha_1 - \alpha_r = r - 1$ (so $\sigma(\pi)$ is the slope of the first part of π, when $\pi \in T$). Conversely, $\pi \notin T \Rightarrow s(\pi) = \beta_1$ and $\sigma(\pi) = \gamma_1 + l - k$.

The action of the map $\pi \to \pi'$ is dependent on which subset of J it is that π is a member of, the subsets being defined as

$$A_1 := \{\pi : s(\pi) > \sigma(\pi), \alpha_k \neq 0, \beta \neq \emptyset\},$$

$$A_2 := \{\pi : s(\pi) > \sigma(\pi), \alpha_k \neq 0, \beta = \emptyset\},$$

$$A_3 := \{\pi : s(\pi) > \sigma(\pi), \alpha_k = 0, \beta \neq \emptyset\},$$

$$A_4 := \{\pi : s(\pi) > \sigma(\pi), \alpha_k = 0, \beta = \emptyset\},$$

$$B_1 := \{\pi : s(\pi) \leq \sigma(\pi), \beta_l = 1\},$$

$$B_2 := \{\pi : s(\pi) = \sigma(\pi), \beta = \emptyset, \pi \notin T\},$$

$$B_3 := \{\pi : s(\pi) < \sigma(\pi), \beta = \emptyset, \pi \notin T\} \cup \{s(\pi) \leq \sigma(\pi), \beta_l \neq 1, \beta \neq \emptyset\},$$

$$B_4 := \{\pi : s(\pi) \leq \sigma(\pi), \pi \in T\}.$$

The set B_3 is defined as the union of two sets, specifically $B_3 = B_3^* \cup B_3^{**}$ where

$$B_3^* := \{\pi : s(\pi) < \sigma(\pi), \beta = \emptyset, \pi \notin T\}$$

and

$$B_3^{**} := \{\pi : s(\pi) < \sigma(\pi), \beta_l \neq 1, \beta \neq \emptyset\}.$$

Now, it is clear from the above definitions that if

$$A := \{\pi \in J : s(\pi) > \sigma(\pi)\} \quad \text{and} \quad B := \{\pi \in J : s(\pi) \leq \sigma(\pi)\}$$

then $J = A \cup B$ and $A = A_1 \cup A_2 \cup A_3 \cup A_4$. It needs to be shown that B can be likewise decomposed.

Clearly $B = \{\pi \in J : s(\pi) \leq \sigma(\pi), \beta \neq \emptyset\} \cup \{\pi \in J : s(\pi) \leq \sigma(\pi), \beta = \emptyset\}$ and the first of these two sets is $B_1 \cup B_3^{**}$. The second set is $\{\pi \in J : s(\pi) \leq \sigma(\pi), \beta = \emptyset\} = \{\pi \in J : s(\pi) \leq \sigma(\pi), \beta = \emptyset, \pi \notin T\} \cup \{\pi \in J : s(\pi) \leq \sigma(\pi), \beta = \emptyset, \pi \in T\}$ and the first of these two sets is $B_2 \cup B_3^*$. The second set is B_4, because $\pi \in T \Rightarrow \beta = \emptyset$ so the second part of any element in B_4 is empty. Hence $B = B_1 \cup B_2 \cup B_3 \cup B_4$.

It is now possible to define, provisionally at least, an involution $\pi \to \pi'$ on J. Depending on which subset of J it is that $\pi = (\alpha, \beta, \gamma) = ((\alpha_1, ..., \alpha_k), (\beta_1, ..., \beta_l), (\gamma_1, ..., \gamma_m))$ is a member of, π' is defined as:

$$\pi' = \begin{cases} ((\alpha_1 - 1, ..., \alpha_k - 1), (\beta_2 + 1, ..., \beta_l + 1, 1), (\beta_1 - l + k, \gamma_1, ..., \gamma_m)) & \pi \in A_1 \\ ((\alpha_1 - 1, ..., \alpha_k - 1), \emptyset, (k, \gamma_1, ..., \gamma_m)) & \pi \in A_2 \\ ((\alpha_1 - 1, ..., \alpha_{k-1} - 1), (\beta_2 + 1, ..., \beta_l + 1), (\beta_1 - l + k, \gamma_1, ..., \gamma_m)) & \pi \in A_3 \\ ((\alpha_1 - 1, ..., \alpha_\sigma - 1, \alpha_{\sigma+1}, ..., \alpha_k), \emptyset, (\gamma_1, ..., \gamma_m, \sigma)) & \pi \in A_4 \\ ((\alpha_1 + 1, ..., \alpha_k + 1), (\gamma_1 + l - k, \beta_1 - 1, ..., \beta_{l-1} - 1), (\gamma_2, ..., \gamma_m)) & \pi \in B_1 \\ ((\alpha_1 + 1, ..., \alpha_k + 1), \emptyset, (\gamma_2, ..., \gamma_m)) & \pi \in B_2 \\ ((\alpha_1 + 1, ..., \alpha_k + 1, 0), (\gamma_1 + l - k, \beta_1 - 1, ..., \beta_l - 1), (\gamma_2, ..., \gamma_m)) & \pi \in B_3 \\ ((\alpha_1 + 1, ..., \alpha_s + 1, \alpha_{s+1}, ..., \alpha_k), \emptyset, (\gamma_1, ..., \gamma_{m-1})) & \pi \in B_4 \end{cases}$$

The element π' may be written as $\pi' = (\delta, \mu, \tau) = ((\delta_1, ..., \delta_K), (\mu_1, ..., \mu_L), (\tau_1, ..., \tau_M))$. For instance if π is such that $\pi' \notin T$ then $s(\pi') = \mu_1$ and $\sigma(\pi') = \tau_1 + L - K$.

Now, by checking each of the eight cases it is easy to see that whenever it is defined the map $\pi \to \pi'$ is weight preserving, $wt(\pi') = wt(\pi)$. It is also easy to see that $\#(\tau) = \#(\gamma) \pm 1$ so $c(\pi') = c(\pi) \pm 1$ which implies that $\pi \to \pi'$ reverses the sign, $(-1)^{c(\pi')} \neq (-1)^{c(\pi)}$. It is also easy to see that the difference in length between the first and second sequence is unaltered in each of the eight cases, i.e. that $K - L = k - l$ so $d(\pi') = d(\pi)$.

What remains to be checked is that this is actually an involution $\pi'' = \pi$, that $\pi \in A_i$ implies that $\pi' \in B_i$, and to find the exceptional elements of J on which the map is not defined. This can be done by looking at each of the eight cases one at a time, before doing this it is shown that $T = A_4 \cup B_4$.

Note that, since $\pi \in T \Rightarrow \alpha_k \neq 0$ (and $\beta \neq \emptyset$), $T \cap A_1 = \emptyset$. Similarly an element of A_2 has $\alpha_k \neq 0$ so $T \cap A_2 = \emptyset$, and $T \cap A_3 = \emptyset$ because $\beta \neq \emptyset$. Now suppose that, for some $\pi \in A_4$, the element π were to be such that $\beta_1 \leq \gamma_1 + l - k$. Then $\beta_1 \leq \gamma_1 + l - k \Rightarrow \pi \notin T$, so (again, since $\beta_1 \leq \gamma_1 + l - k$) $s(\pi) \leq \sigma(\pi)$, which can't happen for $\pi \in A_4$. So if $\pi \in A_4$ then $\beta_1 > \gamma_1 + l - k$, which together with $\alpha_k = 0$ and $\beta = \emptyset$, implies that $\pi \in T$. Hence $T \cap A = T \cap A_4 = A_4$. Similarly if $\pi \in T$ then $\beta = \emptyset$ so π is not in B_1 or B_3^{**}, and neither is it in B_2 or B_3^*. Hence $T \cap B = T \cap B_4 = B_4$ (because $\pi \in B_4 \Rightarrow \pi \in T \Rightarrow \beta = \emptyset$), so $T = A_4 \cup B_4$.

Now let S be a subset of J having the property that if $\pi \in S$ then $\pi \to \pi'$ is defined (i.e. the map is defined on all elements of S). Let S' be the set defined by $S' := \{\pi' : \pi \in S\}$, i.e. S' is the image of S under the map $\pi \to \pi'$. Now for the eight cases:

Case 1, $\pi \in A_1$: From the definition of A_1, any element π in this set is such that the last entry of its first part is nonzero, i.e. $\alpha_k \neq 0$. Hence for $\pi \in A_1$ either $\alpha = \emptyset$, in which case the first part δ of π' is also empty, or $\alpha_k - 1 \geq 0$ and $\delta \in \mathcal{D}_0$ (because $\alpha \in \mathcal{D}$ and each entry in δ is 1 less than the corresponding entry in α).

For any $\pi \in A_1$ $\beta \neq \emptyset$, and if β is a singleton $\beta = (\beta_1)$ then $\mu = (1) \in \mathcal{D}$. If, on the other hand, β has more than one element then $\beta = (\beta_1, \beta_2, ..., \beta_l)$ where $\beta_1 > \beta_2 > ... > \beta_l$ so $\beta_2 + 1 > \beta_3 + 1 > ... > \beta_l + 1 > 1$ and it follows that $\mu = (\beta_2 + 1, \beta_3 + 1, ..., \beta_l + 1, 1) \in \mathcal{D}$.

For any $\pi \in A_1$, $\pi \notin T$ so $s(\pi) = \beta_1 + l - k$ and $\sigma(\pi) = \gamma_1$. The fact that $s(\pi) > \sigma(\pi)$, together with $\pi \notin T$, implies that $\beta_1 - l + k > \gamma_1$ (which ensures that $\beta_1 - l + k > 0$, even if $\gamma = \emptyset$) and so $\tau = (\beta_1 - l + k, \gamma_1, ..., \gamma_m) \in \mathcal{D}$. Thus $\delta \in \mathcal{D}_0$, $\mu \in \mathcal{D}$ and $\tau \in \mathcal{D}$ and so $\pi' = (\delta, \mu, \tau) \in J$, i.e. for each $\pi \in A_1$, π' is defined. What remains to be established is in which subset of J it is that π' lies.

Clearly $\mu_L = 1$, so $\pi' \notin T$, and $\beta_1 > \beta_2$ so $\beta_2 + 1 \leq \beta_1$ and so (since, as always is the case, $k - l = K - L$) $\beta_2 + 1 \leq (\beta_1 - l + k) + L - K$ which is to say that $\mu_1 \leq \tau_1 + L - K$. It follows that, in this case, $s(\pi') \leq \sigma(\pi')$ so $\pi' \in B_1$, i.e. $A_1' \subseteq B_1$, and so that $\pi'' = \pi$.

Case 2, $\pi \in A_2$: Firstly note that, for any $\pi \in A_2$, the first part is nonempty. For $\pi \in A_2 \Rightarrow \pi \notin T$, so $s(\pi) = \beta_1$ and $\sigma(\pi) = \gamma_1 + l - k$. Thus $s(\pi) > \sigma(\pi)$ implies that $\beta_1 > \gamma_1 + l - k$, which is $k > \gamma_1$ (since $\beta = \emptyset$, for $\pi \in A_2$). If there were to be an element $\pi \in A_2$ with $\alpha = \emptyset$, so $k = 0$, then $k > \gamma_1$ would imply $0 > \gamma_1$. This can't happen as $\gamma_1 \geq 0$ (with equality if and only if $\gamma = \emptyset$), so any element of A_2 has nonempty first part.

Now, $\alpha = (\alpha_1, ..., \alpha_k) \in \mathcal{D}$ (but not $\mathcal{D}_0$, since $\alpha_k \neq 0$) and so $\delta = (\alpha_1 - 1, ..., \alpha_k - 1) \in \mathcal{D}_O$. Clearly $\mu = \emptyset$ and since $k > \gamma_1$, $\tau = (k, \gamma_1, ..., \gamma_m) \in \mathcal{D}$ (as in case 1, if $\gamma = \emptyset$ then $k > \gamma_1 \Rightarrow k > 0$ and so $\tau = (k) \in \mathcal{D}$). Hence, for any $\pi \in A_2$, π' is defined.

Clearly $\mu = \emptyset$ and $K = k$ (the length of the first sequence is unaltered). So, since $\tau_1 = k$, $\mu_1 = \tau_1 + L - K$ (because $\mu_1 = 0$ and $L = 0$). The fact that $\mu_1 = \tau_1 + L - K$ ensures that $\pi' \notin T$, so $\mu_1 = \tau_1 + L - K \Rightarrow s(\pi') = \sigma(\pi')$, and $\pi' \in B_2$. It is clear that $A_2' \subseteq B_2$, and also that $\pi'' = \pi$.

Case 3i, $\pi \in A_3$ and $\beta = (\beta_1)$: It is necessary to check that $\pi' = ((\alpha_1 - 1, ..., \alpha_{k-1} - 1), \emptyset, (\beta_1 - l + k, \gamma_1, ..., \gamma_m))$ is actually defined (i.e. π' is in J) for all such elements π:

Now $\pi \in A_3 \Rightarrow \alpha_k = 0 \Rightarrow \alpha_{k-1} - 1 \geq 0$ (note that $\alpha_k = 0$ ensures that $\alpha \neq \emptyset$, so there is no problem in removing the last entry from α).

Since $\pi \in A_3$, it follows that $\pi \notin T$. Thus $s(\pi) = \beta_1$ and $\sigma(\pi) = \gamma_1 + l - k$. Hence $s(\pi) > \sigma(\pi) \Rightarrow \beta_1 - l + k > \gamma_1$ (as in case 1, this ensures that if $\gamma = \emptyset$ then $\beta_1 - l + k > 0$ and so $(\beta_1 - l + k) \in \mathcal{D}$). When $\gamma \neq \emptyset$ this ensures that $\tau = (\beta_1 - l + k, \gamma_1, ..., \gamma_m) \in \mathcal{D}$.

Thus for all $\pi \in A_3$ with β a singleton, the map $\pi \to \pi'$ is defined. Now it is shown that for such π, $\pi' \in B_3^*$:

Clearly $\beta_1 > 0$ so $0 < (\beta_1 - l + k) + L - K$ which, since $\pi \notin T$, means that $s(\pi') = \mu_1 = 0 < \tau_1 + L - K = \sigma(\pi')$, hence $s(\pi') < \sigma(\pi')$. Clearly $\mu = \emptyset$ (and $\pi \notin T$) so $\pi' \in B_3^*$. It is clear that $\pi'' = \pi$.

Note that in the above case β has length 1 so $l = 1$.

Case 3ii, $\pi \in A_3$ and $\beta \neq (\beta_1)$: As in case 3i, $\delta = (\alpha_1 - 1, ..., \alpha_{k-1} - 1) \in \mathcal{D}_0$ and δ is always defined (because, again, $\alpha \neq \emptyset$). Also as in case 3i, $\tau = (\beta_1 - l + k, \gamma_1, ..., \gamma_m)$ is a partition in to distinct parts, even if $\gamma = \emptyset$.

Since, in this case, $\beta = (\beta_1, ..., \beta_l)$ and $l \geq 2$, it must be the case that $\mu = (\beta_2 + 1, ..., \beta_l + 1) \in \mathcal{D}$ and that $\mu \neq \emptyset$ so $\pi' \notin T$. Now $\mu_L = \beta_l + 1 > 1$ and $\pi \notin T$ implies that $s(\pi') = \mu_1 = \beta_2 + 1$ and that $\sigma(\pi') = \tau_1 + L - K = \beta_1 - l + k + L - K = \beta_1$. Since $\beta_1 > \beta_2$, it follows that $\beta_2 + 1 \leq \beta_1$ and so $s(\pi') \leq \sigma(\pi')$. The last inequality, together with $\mu_L > 1$, implies that $\pi' \in B_3^{**}$.

Hence, any $\pi \in A_3$ with two or more entries in the second part is such that π' is defined and an element of B_3^{**}. It is clear that $\pi'' = \pi$.

Hence for any $\pi \in A_3$, it is the case that $\pi' \in B_3$ so $A_3' \subseteq B_3$, and that $\pi'' = \pi$.

Case 4i, $\pi \in A_4$ and $\gamma = \emptyset$: An element π in the set A_4 for which $\gamma = \emptyset$ must be of the form $\pi = ((\alpha_1, \alpha_2, ..., 0), \emptyset, \emptyset)$. Conversely, any element which is of this form has $s(\pi) = \infty > \sigma(\pi)$ and is thus in A_4.

Now any such element π of A_4 has $\sigma(\pi) \leq k$, because the slope cannot exceed the number of parts. For π such that $\sigma = \sigma(\pi) < k$ it is the case that $\pi' = ((\alpha_1 - 1, ..., \alpha_\sigma - 1, \alpha_{\sigma+1}, ..., \alpha_k), \emptyset, (\sigma))$. Note that $0 > \sigma(\pi) - k = \sigma(\pi) + l - k$ and so, since $\mu_1 = 0$ and $\tau_1 = \sigma(\pi)$, $\mu_1 > \tau_1 + L - K$. Thus $\mu_1 > \tau_1 + L - K$, together with $\delta_K = \alpha_k = 0$ and $\mu = \emptyset$, gives $\pi' \in T$.

Now $s(\pi') = \tau_M = \sigma = \sigma(\pi)$ which is the slope of α. The slope of α is less than or equal to the slope of δ which is $\sigma(\pi')$. Thus $s(\pi') \leq \sigma(\pi')$, which together with $\pi \in T$ implies that $\pi' \in B_4$. For such π it is clear that $\pi'' = \pi$.

If, on the other hand such an element π of A_4 is such that the slope of the first part is equal to its length, then $\pi = ((k, k-1, ..., 1, 0), \emptyset, \emptyset)$ for some $k \geq 0$. Clearly the map $\pi \to \pi'$ is not defined on these elements (note that the sequence $(k, k-1, ..., 1, 0)$ has length $k+1$).

Case 4ii, $\pi \in A_4$ and $\gamma \neq \emptyset$: For such π, it is the case that $s(\pi) = \gamma_m$ and that $\sigma = \sigma(\pi)$ is the slope of α. Since $s(\pi) > \sigma(\pi)$, it follows that $\tau = (\gamma_1, ..., \gamma_m, \sigma) \in \mathcal{D}$. Note that there are at least two elements in τ, as opposed to case 4i above where τ was a singleton.

It appears that there is a problem, if $\sigma(\pi) = k$ i.e. if the slope of α were to equal its length then, for since $\alpha_k = 0$, it would follow that $\delta_K = -1$. It is necessary to investigate whether there are any elements for which this happens:

Suppose π is such that, for some $k > 0$, $\pi = ((k-1, k-2, ..., 1, 0), \emptyset, \gamma) \in A_4$ (and $\gamma = (\gamma, ..., \gamma_m) \neq \emptyset$). Since $\pi \in A_4 \Rightarrow \pi \in T$, it must be the case that $\beta_1 > \gamma_1 + l - k$ which, as $\beta = \emptyset$, is to say that $k > \gamma_1$. Clearly $\gamma_1 \geq \gamma_m$ (with equality if and only if $m = 1$). But $s(\pi) > \sigma(\pi) \Rightarrow \gamma_m > k$. So this would give $k > \gamma_1 \geq \gamma_m > k$, which implies the contradiction $k > k$. Thus no such memebers of A_4 exist , i.e. if $\pi \in A_4$ is such that $\gamma \neq \emptyset$ then there must be some entry in α which exceeds by more than 1 the following entry (the slope is is less than the length).

The fact, in this case, $\sigma(\pi) < k$ (as has just been shown) ensures that $\delta_K = \alpha_k = 0$. Clearly $\mu = \emptyset$. As was shown above, $k > \gamma_1$. Thus $0 > \gamma_1 + 0 - k$ which is to say that $\mu_1 > \tau_1 + L - K$ and it follows that $\pi' \in T$. As in case 4i, the slope of π' is at least that of π and so $\pi' \in B_4$, and it is easy to see that $\pi'' = \pi$. As stated earlier, an element $\pi \in A_4$ having $\gamma = \emptyset$ maps to an element $\pi' \in B_4$ for which $\tau = (\tau_1)$ is a singleton, and an element $\pi \in A_4$ having $\gamma \neq \emptyset$ maps to an element $\pi' \in B_4$ for which there are at least two entries in τ.

Case 5i, $\pi \in B_1$ and $\gamma = \emptyset$: The action of the map $\pi \dashrightarrow \pi'$ involves, for $\pi \in B_1$, the removal of the first entry in γ. Thus the map map is not defined for $\pi \in B_1$ when $\gamma = \emptyset$. Thus, it is necessary to estabish which elements of B_1 have third part empty:

If $\pi \in B_1$ has $\gamma = \emptyset$ then $s(\pi) \leq \sigma(\pi) \Rightarrow \beta_1 \leq l - k$, as $\gamma_1 = 0$. Now since β is a partition into distinct parts, it follows that its first part is at least the number of parts: $\beta_1 \geq l$. Thus problematic elements of B_1 satisfy both $\beta_1 \leq l - k$ and $\beta_1 \geq l$, i.e. are such that $k = 0$ so $\alpha = \emptyset$. This implies that $\beta_1 = l$, which happens if and only if the number of entries in β is the number of parts, $\beta = (l, l-1, ..., 2, 1)$ so $\pi = (\emptyset, (l, l-1, ..., 2, 1), \emptyset)$ for l a positive integer. It is clear that the map $\pi \dashrightarrow \pi'$ is not defined on such elements.

Case 5ii, $\pi \in B_1$ and $\gamma \neq \emptyset$: Firstly $(\alpha_1, ..., \alpha_k) \in \mathcal{D}_0$ so either $\alpha = \emptyset$, in which case $\delta = \emptyset$, or $\alpha = (\alpha_1, ..., \alpha_k) \neq \emptyset$ and $\delta = (\alpha_1 + 1, ..., \alpha_k + 1)$. Either way $\delta \in \mathcal{D}_0$ (and the last part of δ is nonzero).

Since $\pi \notin T$, it follows that $s(\pi) = \beta_1$ and $\sigma(\pi) = \gamma_1 + l - k$. Now $s(\pi) \leq \sigma(\pi)$ implies $\gamma_1 + l - k > \beta_1 - 1$ and $\mu \in \mathcal{D}$. Note that $\gamma_1 + l - k > \beta_1 - 1$ ensures that $\gamma_1 + l - k > 0$ and that if the second part of π is a singleton, $\beta = (1)$ then $\mu = (\gamma_1 + l - k)$. Clearly μ is nonempty.

Now $\tau = (\gamma_2, ..., \gamma_m) \in \mathcal{D}$ (so τ is empty if and only if γ is a singleton).

Thus for each $\pi \in B_1$ which has $\gamma \neq \emptyset$, π' exists and has $\delta_K \neq 0$ and $\mu \neq \emptyset$ (either of which imply $\pi' \notin T$). Now, for such π, $\gamma_1 > \gamma_2$ (if γ is a singleton then $\gamma_2 = 0$) so $\gamma_1 + l - k > \gamma_2 + L - K \Rightarrow \mu_1 > \tau_1 + L - K$ which, since $\pi' \notin T$, implies $s(\pi') > \sigma(\pi')$. Thus $\pi' \in A_1$, and it is clear that $\pi'' = \pi$.

Case 6, $\pi \in B_2$: Since $\pi \notin T$, it follows that $s(\pi) = \beta_1$ and $\sigma(\pi) = \gamma_1 + l - k$. Thus $k = \gamma_1$ (since $\beta = \emptyset$). Now if $\gamma = (\gamma_1, ..., \gamma_m) \neq \emptyset$ then $\delta = (\alpha_1 + 1, ..., \alpha_k + 1) \in \mathcal{D}_0$ and the last entry, $\delta_K = \alpha_k + 1 > 0$, clearly $\mu = \emptyset$ and $\tau = (\gamma_2, ..., \gamma_m) \in \mathcal{D}$. Hence the map $\pi \dashrightarrow \pi'$ is defined and $\pi' \in A_2$. It is clear that $\pi'' = \pi$.

Suppose, on the other hand, that an element $\pi \in B_2$ is such that the third part $\gamma = \emptyset$. Then, as above, $k = \gamma_1$ so $k = 0$, i.e. $\alpha = \emptyset$. Since the other other two parts are empty too, it follows that the only such element of B_2 is $\pi = (\emptyset, \emptyset, \emptyset)$ and that this is the only element of B_2 for which the map $\pi \dashrightarrow \pi'$ is undefined.

Case 7i, $\pi \in B_3^*$: Firstly $\pi \notin T$, so $s(\pi) = \beta_1$ and $\sigma(\pi) = \gamma_1 + l - k$. Since $s(\pi) < \sigma(\pi)$, it follows that $\beta_1 < \gamma_1 + l - k$, and since $\beta = \emptyset$ (so $\beta_1 = 0$ and $l = 0$) this implies that $\gamma_1 - k > 0$.

Now it is impossible for an element $\pi \in B_3^*$ to have $\gamma = \emptyset$. For, as shown in the remarks at the start of Case 4i, any element $\pi \in J$ of the form $\pi = ((\alpha_1, ..., 0), \emptyset, \emptyset)$ has $s(\pi) = \infty$ and thus $s(\pi) > \sigma(\pi)$ (in fact it was shown that such elements are necessarily in A_4). Hence $\pi \in B_3^* \Rightarrow \gamma \neq \emptyset$. Thus there is no problem in removing the first entry in the third part, for any $\pi \in B_3^*$.

Thus for any such π the map is defined, specifically $\pi \dashrightarrow \pi'$ where, $\pi' = ((\alpha_1+1,...,\alpha_k+1,0),(\gamma_1-k),(\gamma_2,...,\gamma_m))$. Clearly $\delta_K = 0$. It also follows that $\mu = (\gamma_1 - k) \neq \emptyset$, so $\pi' \notin T$. Thus $s(\pi') = \gamma_1 - k = \gamma_1 + l - k$ and $\sigma(\pi') = \gamma_2 + L - K$ (as in case 5ii, if $\gamma = (\gamma_1)$ then $\gamma_2 = 0$). Since $\gamma_1 > \gamma_2$, it follows that $\pi \in A_3$ and it is clear that $\pi'' = \pi$.

Case 7ii, $\pi \in B_3^{**}$: For any $\pi \in B_3$, the map $\pi \dashrightarrow \pi'$ involves removing the first entry from γ, as happened in case 5i (and case 6). This is a problem if $\gamma = \emptyset$. It is thus necessary to establish whether there are any elements π of B_3^{**} which have third part empty:

Suppose $\pi \in B_3^{**}$ is such that $\gamma = \emptyset$. Then $\pi \in T$ so $s(\pi) = \beta_1$ and $\sigma(\pi) = \gamma_1 + l - k = l - k$. Thus the fact that $s(\pi) \leq \sigma(\pi)$ ensurses that $\beta_1 \leq l - k$. As noted in case 5i, a partition into distinct parts has first part no less than the number of parts, $\beta_1 \geq l$ but since $\pi \in B_3^{**} \Rightarrow \beta_l > 1$, it follows that in this case $\beta_1 > l$. Thus an element of B_3^{**} having $\gamma = \emptyset$ must satisfy both $\beta_1 \leq l - k$ and $\beta_1 > l$, which is impossible. Thus $\pi \in B_3^{**} \Rightarrow \gamma \neq \emptyset$.

It remains to investigate what happens when $\pi \in B_3^{**}$ and $\gamma \neq \emptyset$:

Clearly $\delta = (\alpha_1+1,...,\alpha_k+1,0) \in \mathcal{D}_0$, and so $\delta_K = 0$. Since $\pi \notin T$, it follows that $s(\pi) = \beta_1$ and $\sigma(\pi) = \gamma_1 + l - k$. Thus $s(\pi) \leq \sigma(\pi) \Rightarrow \gamma_1 + l - k > \beta_1 - 1$. From the definition of B_3^{**}, $\beta_1 > 1$ and so $\mu = (\gamma_1 + l - k, \beta_1 - 1, ..., \beta_l - 1) \in \mathcal{D}$. Clearly $\mu \neq \emptyset$, so $\pi' \notin T$. Note that third part of π' is empty if and only if $\gamma = (\gamma_1)$.

Thus π' is defined for all $\pi \in B_3^{**}$. For these such π, since $\pi' \notin T$, it follows that $s(\pi') = \mu_1 = \gamma_1 + l - k$ and $\sigma(\pi) = \tau_1 = \gamma_2 + L - K$ (as usual $\gamma = \emptyset \Rightarrow \gamma_2 = 0$). Clearly $\gamma_1 > \gamma_2$ and so $s(\pi') > \sigma(\pi')$. Hence for these such π, $\pi' \in A_3$ and $\pi'' = \pi$.

Looking at case 7i and case 7ii together, it is seen that for any $\pi \in B_3$, π' is defined and is an element of A_3. Hence $B_3' \subseteq A_3$, and it is clear that $\pi'' = \pi$.

Case 8, $\pi \in B_4$: Firstly, no element $\pi \in B_4$ has $\gamma = \emptyset$. For if there were such an element with $\gamma = \emptyset$ then the fact that $\pi \in T$ would imply that $s(\pi) = \infty > \sigma(\pi)$ (this was explained in the remarks in case 4i, and also in case 7i).

Since any $\pi \in B_4$ is an element of T and has $\gamma \neq \emptyset$, it follows that $s = s(\pi) = \gamma_m$ and $\sigma(\pi)$ is the slope of α. Since $\pi \in T$, it must be the case that $\beta_1 > \gamma_1 + l - k$. Hence (since $\beta = \emptyset$) $k > \gamma_1$, and clearly $\gamma_1 \geq \gamma_m$ (with equality if and only if $m = 1$). Thus $k > \gamma_m$.

So $\pi' = ((\alpha_1+1,...,\alpha_s+1,\alpha_{s+1},...,\alpha_k),\emptyset,(\gamma_1,...,\gamma_{m-1}))$, i.e. the map $\pi \dashrightarrow \pi'$ is defined for all $\pi \in B_4$.

Clearly the length of the first part of π is left unchanged, i.e. δ has the same number of parts as α does. Hence $k = K$ and this, together with $\tau_1 = \gamma_1$ (unless $\gamma = (\gamma_1)$) implies that $K > \tau_1$. If, on the other hand, $\gamma = (\gamma_1)$ then $\tau_1 = 0$, so clearly $K > \tau_1$. Either way $K > \gamma_1 \Rightarrow 0 > \tau_1 + 0 - K$ so $\mu_1 > \tau_1 + L - K$ (because $\mu = \emptyset$). Now $k > \gamma_m$ means that the last element of α is left unchanged, so $\delta_K = \alpha_k = 0$. Thus $\pi' \in T$. All that remains to do is show that $s(\pi') > \sigma(\pi')$:

Now if π' has last part empty, i.e. $\tau = \emptyset$, then (since $\pi' \in T$) $s(\pi') = \infty > \sigma(\pi')$. If, on the other hand, $\tau \neq \emptyset$ then $s(\pi') = \tau_M = \gamma_{m-1}$ and $\sigma(\pi')$ is the slope of δ, which is $s = \gamma_m$. Thus $\gamma_{m-1} > \gamma_m \Rightarrow s(\pi') > \sigma(\pi')$.

Thus, for all $\pi \in B_4$, π' is defined and since $s(\pi) > \sigma(\pi')$, $\delta_K = 0$ and $\mu = \emptyset$ it follows that $\pi' \in A_4$. Hence $B_4' \subseteq A_4$ and it is clear that $\pi'' = \pi$.

Looking at the above eight cases it can be seen that whenever $\pi \in J$ is such that π' is defined then so is π'' and in fact $\pi'' = \pi$.

From case 3 it is seen that $A_3' \subseteq B_3$ and from case 7 that $B_3' \subseteq A_3$. This, together with $\pi'' = \pi$, implies that $A_3' = B_3$ and $B' = A_3$. This can't be done for the other three pairs of sets, because B_1', B_2' and A_4' are not defined (since they contain elements on for which the map is not defined). To remedy this, the following sets are now defined:

$$X_+ := \{\pi \in J : \pi = ((k, k-1, ..., 1, 0), \emptyset, \emptyset), k \geq 0\},$$

$$X_0 := \{(\emptyset, \emptyset, \emptyset)\},$$

$$X_- := \{\pi \in J : \pi = (\emptyset, (l, l-1, ..., 2, 1), \emptyset), l \geq 1\}.$$

Now, given that $\pi'' = \pi$ (whenever π' is defined), it follows from cases 1 and 5 that $A_1' = B_1 \setminus X_-$ and that $(B_1 \setminus X_-)' = A_1$.

Similarly, again given that $\pi'' = \pi$ (whenever π' is defined), it follows from cases 2 and 6 that $A_3' = B_3 \setminus X_0$ and that $(B_3 \setminus X_0)' = A_3$.

Finally, given that $\pi'' = \pi$ (whenever π' is defined), it follows from cases 4 and 8 that $(A_4 \setminus X_+)' = B_4$ and that $B_4' = A_4 \setminus X_+$.

What all this means is that if $\pi \notin X := X_+ \cup X_0 \cup X_-$ then π and π' can be paired off with each other. Recall that the map $\pi \to \pi'$ is weight preserving, i.e. $wt(\pi') = wt(\pi)$, and also $d(\pi') = d(\pi)$ but $(-1)^{c(\pi')} \neq (-1)^{c(\pi)}$. Thus the map preserves everything apart from the sign, so the contribution made by a given π in the sum

$$\sum_{\pi \in J} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)}$$

is cancelled out by the contribution made by π', and so the above sum reduces to a sum over X. To be precise,

$$\sum_{\pi \in J} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)} = \sum_{\pi \in J \setminus X} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)} + \sum_{\pi \in X} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)}$$

$$= \sum_{\substack{\pi \in J \setminus X \\ c(\pi) \text{even}}} z^{d(\pi)} q^{wt(\pi)} - \sum_{\substack{\pi \in J \setminus X \\ c(\pi) \text{odd}}} z^{d(\pi)} q^{wt(\pi)} + \sum_{\pi \in X} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)}$$

and since, for $\pi \in J \setminus X$, $c(\pi)$ is even if and only $c(\pi')$ is odd, the first two sums cancel out and so

$$\sum_{\pi \in J} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)} = \sum_{\pi \in X} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)}$$

$$= \sum_{\pi \in X_0 \cup X_+ \cup X_-} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)}$$

$$= \sum_{\pi \in X_0} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)} + \sum_{\pi \in X_+} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)} + \sum_{\pi \in X_-} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)}$$

which, since $c(\pi) = 0$ for all $\pi \in X$, becomes

$$= \sum_{\pi \in X_0} z^{d(\pi)} q^{wt(\pi)} + \sum_{\pi \in X_+} z^{d(\pi)} q^{wt(\pi)} + \sum_{\pi \in X_-} z^{d(\pi)} q^{wt(\pi)}$$

$$= 1 + \sum_{k \geq 0} z^{k+1} q^{k+k-1+\ldots+1+0} + \sum_{l>1} z^{-l} q^{l+l-1+\ldots+2+1}$$

$$= 1 + \sum_{k \geq 0} z^{k+1} q^{\frac{k^2+k}{2}} + \sum_{l>1} z^{-l} q^{\frac{l^2+l}{2}}$$

$$= 1 + \sum_{k \geq 0} z^{k+1} q^{\frac{(k+1)^2-(k+1)}{2}} + \sum_{l>1} z^{-l} q^{\frac{(-l)^2-(-l)}{2}}$$

$$= 1 + \sum_{n>0} z^n q^{\frac{n^2-n}{2}} + \sum_{n<0} z^n q^{\frac{n^2-n}{2}}$$

and so

$$\sum_{\pi \in J} (-1)^{c(\pi)} z^{d(\pi)} q^{wt(\pi)} = \sum_{n>0} z^n q^{\frac{n^2-n}{2}}.$$

This proves (4.2), and so (4.1).

Chapter 5

Ranks and biranks

5.1 A generalisation of an identity of Fine

The following new identity is a generalisation of an identity due to Fine:

$$\sum_{n\geq 0}\frac{(ax^nq;q)_n t^n}{(q;q)_n}=\sum_{m\geq 0}\frac{(-1)^m a^m t^m x^{m^2} q^{\frac{m^2+m}{2}}}{(q;q)_m (tx^m;q)_\infty} \tag{5.1}$$

Convergence is ensured by stipulating that $|a|\leq 1$, $|t|<1$, $|x|<1$ (and, as usual for infinite series, $|q|<1$).

This identity is now proved by showing that the coefficient of t^k on the right is the same as that on the left:

Firstly the coefficient of t^k on the right is the coefficient of t^k in the finite sum

$$\sum_{m=0}^{k}\frac{(-1)^m a^m t^m x^{m^2} q^{\frac{m^2+m}{2}}}{(q;q)_m (tx^m;q)_\infty}.$$

Now, putting $n\to\infty$ and $z=tx^m$ in the identity of Rothe, (1.10), gives

$$\frac{1}{(tx^m;q)_\infty}=\sum_{l\geq 0}\frac{t^l x^{ml}}{(q;q)_l}$$

and so the coefficient of t^k on the right of (5.1) is the coefficient of t^k in

$$\sum_{m=0}^{k}\left(\frac{(-1)^m a^m t^m x^{m^2} q^{\frac{m^2+m}{2}}}{(q;q)_m}\sum_{l\geq 0}\frac{t^l x^{ml}}{(q;q)_l}\right)=\sum_{m=0}^{k}\left(\frac{(-1)^m a^m x^{m^2} q^{\frac{m^2+m}{2}}}{(q;q)_m}\sum_{l\geq m}\frac{t^l x^{m(l-m)}}{(q;q)_{l-m}}\right)$$

and the coefficient of t^k in this is

$$\sum_{m=0}^{k}\left(\frac{(-1)^m a^m x^{m^2} q^{\frac{m^2+m}{2}}}{(q;q)_m}.\frac{x^{m(k-m)}}{(q;q)_{k-m}}\right)=\sum_{m=0}^{k}\frac{(-1)^m a^m x^{km} q^{\frac{m^2+m}{2}}}{(q;q)_m (q;q)_{k-m}}$$

which, in terms of q-binomial coefficients, is

$$\frac{1}{(q;q)_k}\sum_{m=0}^{k}(-1)^m a^m x^{km} q^{\frac{m^2+m}{2}}\begin{bmatrix}k\\m\end{bmatrix}. \tag{5.2}$$

Finally, by putting $z = ax^k q$ in identity (1.9), it can be seen that the coefficient of t^k on the left of (5.1) is

$$\frac{(ax^k q;q)_k}{(q;q)_k} = \frac{1}{(q;q)_k}\sum_{m=0}^{k}(-1)^m a^m x^{km} q^{\frac{m^2+m}{2}}\begin{bmatrix}k\\m\end{bmatrix}. \tag{5.3}$$

Identity (5.3) shows that the coefficient on the right of (5.1) equals the coefficient of t^k on the left of (5.1), proving (5.1) as required.

5.1.1 The above identity and partitions

When $x = q$ is put into in (5.1) the resulting special case is

$$\sum_{n\geq 0}\frac{(aq^{n+1};q)_n t^n}{(q;q)_n} = \sum_{m\geq 0}\frac{(-1)^m a^m t^m q^{\frac{3m^2+m}{2}}}{(q;q)_m(tq^m;q)_\infty}$$

which is due to Fine, it is identity 25.94 in [11]. Fine uses this identity to find the generating function for the rank. This is done here too, but in more detail than in Fine's book. Before doing this it is of course necessary to define the rank of a partition, this is done in the next section, but first it should be noted that putting $t = q^2$ into the above identity gives

$$\sum_{n\geq 0}\frac{(aq^{n+1};q)_n q^{2n}}{(q;q)_n} = \frac{1}{(q;q)_\infty}\sum_{m\geq 0}(-1)^m a^m q^{\frac{3m^2+5m}{2}}(1-q^{m+1})$$

and for h a nonnegative integer putting $a = q^h$ and rearranging gives

$$\sum_{n\geq 0}\begin{bmatrix}2n+h\\n\end{bmatrix}q^{2n} = \frac{1}{(q;q)_\infty}\sum_{m\geq 0}(-1)^m q^{\frac{3m^2+5m}{2}+hm}(1-q^{m+1}). \tag{5.4}$$

5.2 The rank generating function

The partition function, $p(n)$, is known to satisfy certain congruences: including (1.19), (1.20) and (1.21) above which, to recap, state

$$p(5m+4) \equiv 0 \bmod 5,$$

$$p(7m+5) \equiv 0 \bmod 7,$$

$$p(11m+6) \equiv 0 \bmod 11.$$

These can be proved by considering the generating function, $(q;q)_\infty^{-1}$ of $p(n)$, how to do this for (1.19) was outlined in section 1.3.2. Now proving that $p(5m+4) \equiv 0 \bmod 5$

by invoking the generating function is an instance of a non-combinatorial proof: It has been established that 5 divides $p(4), p(9), p(14), \ldots$ but the proof does not describe how to write the set of partitions of weight 4, 9 or 14 etc as the disjoint union of five equinumerous subsets.

Suppose that there were to be a map $g : \mathcal{P} \to \mathbb{Z}_5$ with the property that for $n \equiv 4 \bmod 5$, it were the case that

$$x \in \mathbb{Z}_5 \Rightarrow \#\{\lambda \in \mathcal{P} : wt(\lambda) = n \text{ and } g(\lambda) = x\} = \frac{1}{5}p(n)$$

then there would indeed be five equinumerous sets.

The first person to actually invent a statistic with the properties described above was Dyson. He defined, in [9], the *rank* of a partition as the first part minus the number of parts:

$$rk(\lambda) := \lambda_1 - \#(\lambda) \tag{5.5}$$

or, equivalently, $rk(\lambda) := \lambda_1 - k$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. The rank of the empty partition is taken to be zero, $rk(\emptyset) = 0$. The rank is therefore a map from $\mathcal{P}$ to $\mathbb{Z}$, not $\mathbb{Z}_5$, but reduction mod 5 remedies this.

To do this it helps to introduce the following standard notation:

$$N(m, n) := \#\{\lambda \in \mathcal{P} : wt(\lambda) = n \text{ and } rk(\lambda) = m\} \tag{5.6}$$

and

$$N(r, m, n) := \#\{\lambda \in \mathcal{P} : wt(\lambda) = n \text{ and } rk(\lambda) \equiv r \bmod m\}. \tag{5.7}$$

Thus $N(m, n)$ is, by definition, the number of partitions of weight n having rank m and $N(r, m, n)$ is defined to be the the number of partitions of weight n whose rank is congruent to $r \bmod m$.

If $r' \equiv r \bmod m$ then $N(r', m, n) = N(r, m, n)$, so it suffices to find $N(r, m, n)$ for $0 \leq r < m$. By considering the dual, λ' of a given partition λ, it is easy to see that $rk(\lambda') = -rk(\lambda)$. Hence $N(m, n) = N(-m, n)$ and so $N(m - r, m, n) = N(r, m, n)$. Hence it transpires that it suffices to determine $N(r, m, n)$ in the range $0 \leq r \leq \lfloor m/2 \rfloor$.

It is clear that

$$N(r, m, n) = \sum_{t \equiv r \bmod m} N(t, n)$$

and that

$$\begin{aligned} p(n) &= \sum_{m \in \mathbb{Z}} N(m, n) \\ &= \sum_{m=-n+1}^{n-1} N(m, n) \end{aligned}$$

and also that

$$p(n) = \sum_{r=0}^{m-1} N(r, m, n)$$

As an example, consider the partitions of weight 9: There are six such partitions having rank congruent to 0 mod 5;

$$(2,2,1,1,1,1,1), \quad (3,3,3), \quad (4,3,1,1),$$

$$(4,2,2,1), \quad (5,1,1,1,1), \quad (7,2).$$

Hence $N(0,5,9)=6$. Likewise $N(1,5,9)=6$, because there are six partitions of weight 9 having rank congruent to 1 mod 5;

$$(3,1,1,1,1,1,1), \quad (2,2,2,1,1,1), \quad (4,3,2),$$

$$(4,4,1), \quad (5,2,1,1), \quad (8,1).$$

and so it also follows that $N(4,5,9) = 6$, as the dual of any of the above partitions has rank congruent to 4 mod 5. Finally $N(2,5,9) = 6$ (and so, by looking at the duals, $N(3,5,9)=6$) because of the following six partitions;

$$(1,1,1,1,1,1,1,1,1), \quad (3,2,1,1,1,1), \quad (2,2,2,2,1),$$

$$(5,2,2), \quad (5,3,1), \quad (6,1,1,1).$$

So

$$N(0,5,9)=N(1,5,9)=N(2,5,9).$$

Dyson, noticing equalities such as the one above, conjectured that

$$N(0,5,5m+4)=N(1,5,5m+4)=N(2,5,5m+4) \tag{5.8}$$

which is both a proof, and a combinatorial interpretation of the statement $p(5m+4) \equiv 0 \bmod 5$. Dyson also conjectured that

$$N(0,7,7m+5)=N(1,7,7m+5)=N(2,7,7m+5)=N(3,7,7m+5) \tag{5.9}$$

which is likewise related to the congruence $p(7m+5) \equiv 0 \bmod 7$. Dyson stated his conjectures in [9]. Later they were proved by Atkin and Swinnerton-Dyer, [5].

5.3 The generating function

Let $\mathcal{P}$ denote the set of all partitions. The generating function for the Dyson rank alluded to earlier is

$$\sum_{\lambda\in\mathcal{P}} z^{rk(\lambda)} q^{wt(\lambda)} = \frac{(1-z)}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{n(3n+1)/2}}{1-zq^n}. \tag{5.10}$$

The generating function for the rank was first presented in [5], in a slightly different form to (5.10) above.

Identity (5.10) is now proved by writing the expression on the right as a power series in z, where the coefficients are q-series, and then using (5.4) to show that this is the same as the expression on the left of (5.10).

Now, the expression on the right of (5.10) is

$$\frac{(1-z)}{(q;q)_\infty}\left(\frac{1}{1-z}+\sum_{n\in\mathbb{Z}\setminus\{0\}}(-1)^n\frac{q^{n(3n+1)/2}}{1-zq^n}\right)$$

and the sum in the above expression is

$$\sum_{n\in\mathbb{Z}\setminus\{0\}}(-1)^n\frac{q^{n(3n+1)/2}}{1-zq^n}=\sum_{\substack{n \text{ even}\\(n\neq 0)}}\frac{q^{n(3n+1)/2}}{1-zq^n}-\sum_{n \text{ odd}}\frac{q^{n(3n+1)/2}}{1-zq^n}$$

$$=\sum_{\substack{n>0\\n \text{ even}}}\frac{q^{n(3n+1)/2}}{1-zq^n}+\sum_{\substack{n<0\\n \text{ even}}}\frac{q^{n(3n+1)/2}}{1-zq^n}-\sum_{\substack{n>0\\n \text{ odd}}}\frac{q^{n(3n+1)/2}}{1-zq^n}-\sum_{\substack{n<0\\n \text{ odd}}}\frac{q^{n(3n+1)/2}}{1-zq^n}$$

$$=\sum_{\substack{n>0\\n \text{ even}}}\left(\frac{q^{(3n^2+n)/2}}{1-zq^n}+\frac{q^{(3n^2-n)/2}}{1-zq^{-n}}\right)-\sum_{\substack{n>0\\n \text{ odd}}}\left(\frac{q^{(3n^2+n)/2}}{1-zq^n}+\frac{q^{(3n^2-n)/2}}{1-zq^{-n}}\right)$$

which is equal to

$$\sum_{\substack{n=2m\\m>0}}\left(\frac{q^{(3n^2+n)/2}}{1-zq^n}-z^{-1}\frac{q^{(3n^2+n)/2}}{1-z^{-1}q^n}\right)-\sum_{\substack{n=2m-1\\m>0}}\left(\frac{q^{(3n^2+n)/2}}{1-zq^n}-z^{-1}\frac{q^{(3n^2+n)/2}}{1-z^{-1}q^n}\right)$$

$$=\sum_{m>0}\left(\frac{q^{6m^2+m}}{1-zq^{2m}}-z^{-1}\frac{q^{6m^2+m}}{1-z^{-1}q^{2m}}\right)-\sum_{m>0}\left(\frac{q^{6m^2-5m+1}}{1-zq^{2m-1}}-z^{-1}\frac{q^{6m^2-5m+1}}{1-z^{-1}q^{2m-1}}\right)$$

$$=\left(\sum_{m>0}\left(\sum_{h\geq 0}z^hq^{6m^2+m+2hm}\right)\right)-z^{-1}\left(\sum_{m>0}\left(\sum_{h\geq 0}z^{-h}q^{6m^2+m+2hm}\right)\right)$$

$$-\left(\sum_{m>0}\left(\sum_{h\geq 0}z^hq^{6m^2-5m+1+h(2m-1)}\right)\right)+z^{-1}\left(\sum_{m>0}\left(\sum_{h\geq 0}z^{-h}q^{6m^2-5m+1+h(2m-1)}\right)\right)$$

$$=\left(\sum_{h\geq 0}z^h\left(\sum_{m>0}q^{6m^2+(2h+1)m}\right)\right)-\left(\sum_{h>0}z^{-h}\left(\sum_{m>0}q^{6m^2+(2h-1)m}\right)\right)$$

$$-\left(\sum_{h\geq 0}z^h\left(\sum_{m>0}q^{6m^2+(2h-5)m-h+1}\right)\right)+\left(\sum_{h>0}z^{-h}\left(\sum_{m>0}q^{6m^2+(2h-7)m-h+2}\right)\right).$$

Multiplying through by $1 - z$ gives

$$(1-z) \sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n \frac{q^{n(3n+1)/2}}{1-zq^n}$$

$$= \left(\sum_{h\geq 0} z^h \left(\sum_{m>0} q^{6m^2+(2h+1)m}\right)\right) - \left(\sum_{h>0} z^{-h}\left(\sum_{m>0} q^{6m^2+(2h-1)m}\right)\right)$$

$$-\left(\sum_{h\geq 0} z^h \left(\sum_{m>0} q^{6m^2+(2h-5)m-h+1}\right)\right) + \left(\sum_{h>0} z^{-h}\left(\sum_{m>0} q^{6m^2+(2h-7)m-h+2}\right)\right)$$

$$-\left(\sum_{h\geq 0} z^{h+1} \left(\sum_{m>0} q^{6m^2+(2h+1)m}\right)\right) + \left(\sum_{h>0} z^{-h+1}\left(\sum_{m>0} q^{6m^2+(2h-1)m}\right)\right)$$

$$+\left(\sum_{h\geq 0} z^{h+1} \left(\sum_{m>0} q^{6m^2+(2h-5)m-h+1}\right)\right) - \left(\sum_{h>0} z^{-h+1}\left(\sum_{m>0} q^{6m^2+(2h-7)m-h+2}\right)\right)$$

and so

$$(1-z) \sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{n(3n+1)/2}}{1-zq^n}$$

$$= 1 + \left(\sum_{h\geq 0} z^h \left(\sum_{m>0} q^{6m^2+(2h+1)m}\right)\right) - \left(\sum_{h>0} z^{-h}\left(\sum_{m>0} q^{6m^2+(2h-1)m}\right)\right)$$

$$-\left(\sum_{h\geq 0} z^h \left(\sum_{m>0} q^{6m^2+(2h-5)m-h+1}\right)\right) + \left(\sum_{h>0} z^{-h}\left(\sum_{m>0} q^{6m^2+(2h-7)m-h+2}\right)\right)$$

$$-\left(\sum_{h>0} z^{h} \left(\sum_{m>0} q^{6m^2+(2h-1)m}\right)\right) + \left(\sum_{h\geq 0} z^{-h}\left(\sum_{m>0} q^{6m^2+(2h+1)m}\right)\right)$$

$$+\left(\sum_{h>0} z^{h} \left(\sum_{m>0} q^{6m^2+(2h-7)m-h+2}\right)\right) - \left(\sum_{h\geq 0} z^{-h}\left(\sum_{m>0} q^{6m^2+(2h-5)m-h+1}\right)\right). \quad (5.11)$$

The above expression is clearly invariant under $z \to z^{-1}$. When $z = 1$ the whole thing is equal to 1 and so, dividing by $(q;q)_\infty$ implies that, the left of (5.10) is invariant under $z \to z^{-1}$ (which is to be expected as the rank of each partition is minus the value of the rank of its dual) and equal to $P(q) = (q;q)_\infty^{-1}$ when $z = 1$.

Now, for $h > 0$, the coefficient of z^h in the expression on the right of (5.11) is

$$\sum_{m>0} q^{6m^2+(2h+1)m} - \sum_{m>0} q^{6m^2+(2h-5)m-h+1} - \sum_{m>0} q^{6m^2+(2h-1)m} + \sum_{m>0} q^{6m^2+(2h-7)m-h+2}$$

(note that if the h in the above expression were replaced by $|h|$, then it would also hold for negative h). This is equal to

$$\sum_{m>0} q^{6m^2+(2h-7)m-h+2}(1 - q^{2m-1} - q^{6m+h-2} + q^{8m+h-2})$$

$$= q\sum_{m\geq 0} q^{6m^2+(2h+5)m+h}\left(1-q^{2m+1}-q^{6m+h+4}+q^{8m+h+6}\right)$$

$$= q^{h+1}\left(\sum_{m\geq 0} q^{2mh}.q^{6m^2+5m}\left(1-q^{2m+1}\right)-\sum_{m\geq 0} q^{(2m+1)h}.q^{6m^2+11m+4}\left(1-q^{2m+2}\right)\right)$$

$$= q^{h+1}\left(\sum_{\substack{s=2m\\ m\geq 0}} q^{hs+\frac{3s^2+5s}{2}}\left(1-q^{s+1}\right)-\sum_{\substack{s=2m+1\\ m\geq 0}} q^{hs+\frac{3s^2+5s}{2}}\left(1-q^{s+1}\right)\right)$$

$$= q^{h+1}\sum_{s\geq 0}(-1)^s q^{hs+\frac{3s^2+5s}{2}}\left(1-q^{s+1}\right)$$

and the coefficient of z^0 on the right of (5.11) is $1-2q^2+2q^7+\ldots+(-1)^s 2q^{\frac{3s^2+s}{2}}+\ldots$. It has been shown that the coefficient of z^h in

$$\frac{(1-z)}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n\frac{q^{n(3n+1)/2}}{(1-zq^n)}$$

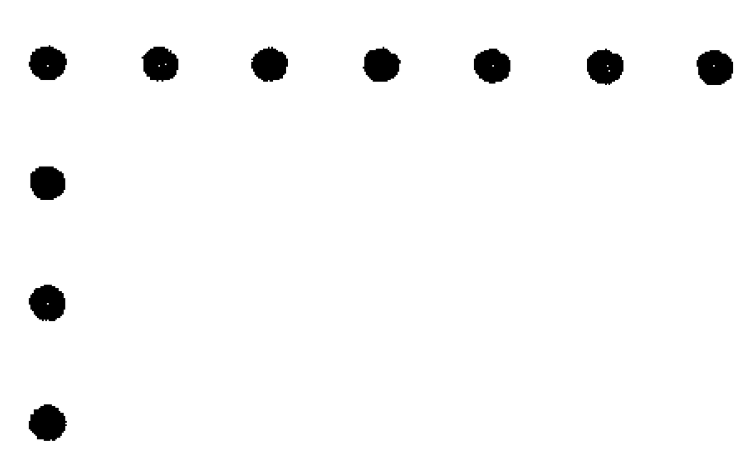
is

$$\frac{q^{h+1}}{(q;q)_\infty}\sum_{s\geq 0}(-1)^s q^{hs+\frac{3s^2+5s}{2}}\left(1-q^{s+1}\right) \quad \text{if } h>0, \tag{5.12}$$

$$\frac{1}{(q;q)_\infty}\left(1+2\sum_{s\geq 0}(-1)^s q^{\frac{3s^2+s}{2}}\right) \quad \text{if } h=0. \tag{5.13}$$

5.4 Ranks

If a partition λ has $rk(\lambda)=h$, where $h>0$ then it must have first part $\lambda_1=n+h+1$ and number of parts $\#(\lambda)=n+1$ parts some $n\geq 0$. Conversely whenever it is the case that, for some fixed h, there exists some n such that $\lambda_1=n+h+1$ and $\#(\lambda)=n+1$ then it must be the case that $rk(\lambda)=h$. Thus the graph of a partition having rank h has $n+h+1$ dots in the top row and $h+1$ dots in the left column, as shown



this leaves and $n+h$ by n rectangle 'inside' that is to be filled with some partition. Thus, by lemma (1.8) summing over all such partitions gives

$$\sum q^{wt(\lambda)} = q^{2n+h+1}\begin{bmatrix}2n+h\\ n\end{bmatrix}$$

and summing over all n gives

$$\sum_{rk(\lambda)=h} q^{wt(\lambda)} = q^{h+1}\sum_{n\geq 0} q^{2n}\begin{bmatrix}2n+h\\ n\end{bmatrix}. \tag{5.14}$$

For partitions having rank 0 there is a problem as the empty partition has not been counted. This is easily remedied; for $h = 0$ identity (5.14) becomes

$$\sum_{rk(\lambda)=0} q^{wt(\lambda)} = 1 + q\sum_{n\geq 0} q^{2n}\begin{bmatrix}2n\\ n\end{bmatrix} \tag{5.15}$$

Now the expression on the right of (5.14) is q^{h+1} multiplied by the expression on the left of (5.4), so for $h > 0$

$$\sum_{rk(\lambda)=h} q^{wt(\lambda)} = \frac{q^{h+1}}{(q;q)_\infty}\sum_{m\geq 0}(-1)^m q^{\frac{3m^2+5m}{2}+hm}(1-q^{m+1}). \tag{5.16}$$

and by (5.12), it is seen that for $h > 0$, it is indeed the case that summing $q^{wt(\lambda)}$ over all λ having rank h is the coefficient of z^h in the expression on the right of (5.10). This being the case, it must also hold for $h < 0$, for it has been shown that this expression is invariant under $z \to z^{-1}$.

It remains to check the case $h = 0$: Because

$$1 + \frac{q}{(q;q)_\infty}\sum_{m\geq 0}(-1)^m q^{\frac{3m^2+5m}{2}}\left(1-q^{m+1}\right) = \frac{1}{(q;q)_\infty}\left(1 + 2\sum_{s\geq 0}(-1)^s q^{\frac{3s^2+s}{2}}\right)$$

it follows that (5.15) together with $h = 0$ in (5.4) and (5.13) imply that $q^{wt(\lambda)}$ summed over all partitions having rank zero is indeed the coefficient of z^0 in (5.10). This proves (5.10).

5.5 The Birank

The rank was invented to explain congruences in the sequence $p_{-1}(n)$, that it does so is seen from (5.8) and (5.9). There are similiar congruences in sequences having the form $p_k(n)$ where $k \neq -1$. In particular, for $x = 2, 3$ or 4,

$$p_{-2}(5n + x) \equiv 0 \bmod 5 \tag{5.17}$$

which, given that $p_{-2}(n)$ is the coefficient of q^n in $(q;q)_\infty^{-2}$, can be proved by an argument similar to that used to prove (1.19).

There is, so far at least, a relative shortage of Dyson-type ranks related to such sequences, but [14] looks at the sequence $p_{-24}(n)$. What follows here is about the sequence $p_{-2}(n)$: a birank is introduced related to the congruences in (5.17). The behaviour of the birank modulo 2, 3, 4, 6 and 8 is investigated, as well as modulo 5 which is perhaps the most interesting case as this gives rise to actual congruences in the sequence $p_{-2}(n)$. A paper on the mod 5 case, which is dealt with in theorem 4 below has recently been published [19].

The *birank* of an ordered pair of partitions, $\pi = (\lambda, \mu)$ is here defined to be the number of parts of the first partition minus the number of parts of the second partition:

$$b(\pi) := \#(\pi) - \#(\mu) \tag{5.18}$$

or, equivalently, $b(\pi) := k - l$ where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ and $\mu = (\mu_1, \mu_2, ..., \mu_l)$.

The following notation is analogous to that for the rank ((5.6) and (5.7)),

$$R(m,n) := \#\{\pi \in \mathcal{P}^2 : wt(\pi) = n \text{ and } b(\pi) = m\}, \tag{5.19}$$

where the weight of the ordered pair $\pi = (\lambda, \mu)$, is defined to be $wt(\pi) := wt(\lambda) + wt(\mu)$, and

$$R(r,m,n) := \#\{\pi \in \mathcal{P}^2 : wt(\pi) = n \text{ and } b(\pi) \equiv r \bmod m\}. \tag{5.20}$$

Clearly $b(\lambda, \mu) = -b(\mu, \lambda)$ and so $R(m,n) = R(-m,n)$ and $R(m-r,m,n) = R(r,m,n)$. In this respect the birank introduced here is similar to the rank of Dyson.

As with the rank, it is necessary to look at the generating function for the birank. Fortunately this is easier than for the rank. It is clear that

$$\sum_{\pi \in \mathcal{P}^2} z^{b(\pi)} q^{wt(\pi)} = \frac{1}{(zq;q)_\infty (z^{-1}q;q)_\infty} \tag{5.21}$$

which is to say that

$$\sum_{m \in \mathbb{Z}} \sum_{n \geq 0} R(m,n) z^m q^n = \frac{1}{(zq;q)_\infty (z^{-1}q;q)_\infty}. \tag{5.22}$$

In the next section, the behaviour of the birank is investigated by substituting suitable values for z into (5.22). The approach is (for a specified value of k) to put $z = \omega$, where ω is some root of unity, usually $\omega = e^{\frac{2\pi i}{k}}$ in which case it follows that

$$\sum_{m \in \mathbb{Z}} \sum_{n \geq 0} R(m,n) \omega^m q^n = R[0,k] + \omega R[1,k] + ... + \omega^{k-1} R[k-1,k] \tag{5.23}$$

where $R[r,k]$ is defined as

$$\begin{aligned} R[r,k] &:= \sum_{n \geq 0} R(r,k,n) q^n \\ &= \sum_{s \equiv_k r} \sum_{n \geq 0} R(s,n) q^n \\ &= \sum_{b(\pi) \equiv_k r} q^{wt(\pi)} \end{aligned}$$

(here $\equiv_k$ means congruent mod k).

5.6 The behaviour of the birank

The following theorems hold for n any natural number, bar the exceptions listed.

Theorem 1:

$$R(0,2,2n) > R(1,2,2n), \quad R(0,2,2n+1) < R(1,2,2n+1). \tag{5.24}$$

Theorem 2:

$$R(0,3,3n) > R(1,3,3n),$$

$$R(0,3,3n+1) < R(1,3,3n+1), \quad R(0,3,3n+2) < R(1,3,3n+2). \tag{5.25}$$

The only exception is $R(0,3,5) = R(1,3,5)$.

Theorem 3:

$$R(0,4,4n) > R(2,4,4n), \quad R(2,4,4n+2) > R(0,4,4n+2),$$

$$R(0,4,2n) > R(1,4,2n), \quad R(2,4,2n) > R(1,4,2n),$$

$$R(0,4,2n+1) = R(2,4,2n+1) < R(1,4,2n+1). \tag{5.26}$$

With two exceptions, $R(0,4,2) = R(1,4,2)$ and $R(0,4,4) = R(2,4,4)$.

Theorem 4:

$$R(0,5,5n) > R(1,5,5n) = R(2,5,5n),$$

$$R(1,5,5n+1) > R(0,5,5n+1) = R(2,5,5n+1),$$

and the following equalities (which imply (5.17)),

$$R(0,5,5n+2) = R(1,5,5n+2) = R(2,5,5n+2),$$

$$R(0,5,5n+3) = R(1,5,5n+3) = R(2,5,5n+3),$$

$$R(0,5,5n+4) = R(1,5,5n+4) = R(2,5,5n+4). \tag{5.27}$$

The only exception is $R(0,5,6) = R(1,5,6)$.

Theorem 5:

$$R(0,6,n) > R(2,6,n),$$

$$R(0,6,2n) > R(3,6,2n), \quad R(0,6,2n+1) < R(3,6,2n+1). \tag{5.28}$$

With exceptions: $n = 1,2,4 \Rightarrow R(0,6,n) = R(2,6,n)$ and $n = 1,3,5,7 \Rightarrow R(0,6,n) = R(3,6,n)$.

Theorem 6:

$$R(0,8,n) > R(4,8,n), \quad R(1,8,n) > R(3,8,n). \tag{5.29}$$

The first inequality failing only at $n = 1$, the second only at $n = 0$.

In the following proofs, repeated use will be made of the triple product identity,

$$[z;q](q;q)_\infty = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2-n}{2}}. \tag{5.30}$$

This is identity (4.2), proved in chapter 4.

The triple product is useful here because it be used to obtain dissections of q-series. A *dissection* of a series, $F(q)$, is an expression of the form

$$F(q) = F_0(q^n) + qF_1(q^n) + \dots + q^{n-1}F_{n-1}(q^n).$$

For example,

$$\frac{1}{1-q} = \frac{1}{1-q^3} + \frac{q}{1-q^3} + \frac{q^2}{1-q^3}$$

is the 3-dissection of $1/(1-q)$.

When looking at dissections of certain q-series, the following notation will prove useful:

$$[a,b,\dots,x;q] := [a;q][b;q]\dots[x;q].$$

Theorem 1.

It is necessary to consider only $R(0,2,n) - R(1,2,n)$. Putting $z=-1$ in (5.22) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)(-1)^m q^n = \frac{1}{(-q;q)^2_\infty}$$

and so

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)(-1)^m q^n = (q;q^2)^2_\infty$$

which is to say that

$$\sum_{m \text{ even}}\sum_{n\geq 0} R(m,n)q^n - \sum_{m \text{ odd}}\sum_{n\geq 0} R(m,n)q^n = (q;q^2)^2_\infty$$

so, by (5.23) (with $k=2$, since $\omega=-1$),

$$R[0,2] - R[1,2] = (q;q^2)^2_\infty. \tag{5.31}$$

Thus it is necessary to show that the coefficient of q^n in $(q;q^2)^2_\infty$ is positive if n is even and negative if n is odd. Now

$$(q;q^2)^2_\infty = [q;q^2]$$

so

$$(q;q^2)^2_\infty = \frac{[q;q^2](q^2;q^2)_\infty}{(q^2;q^2)_\infty}$$

and so, by the triple product identity ($q \to q^2$ in (5.30) and then set $z=q$),

$$(q;q^2)^2_\infty = \frac{1}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}$$

$$= \frac{1}{(q^2;q^2)_\infty}\left(\sum_{n \text{ even}} q^{n^2} - \sum_{n \text{ odd}} q^{n^2}\right)$$

$$= \frac{1}{(q^2;q^2)_\infty}\left(\sum_{\substack{n=2m\\ m\in\mathbb{Z}}} q^{n^2} - \sum_{\substack{n=2m-1\\ m\in\mathbb{Z}}} q^{n^2}\right)$$

$$= \frac{1}{(q^2;q^2)_\infty}\left(\sum_{m\in\mathbb{Z}} q^{(2m)^2} - \sum_{m\in\mathbb{Z}} q^{(2m-1)^2}\right)$$

$$= \frac{1}{(q^2;q^2)_\infty}\left(\sum_{m\in\mathbb{Z}} q^{4m^2} - q\sum_{m\in\mathbb{Z}} q^{4m^2-4m}\right).$$

This is the 2-dissection of $(q;q^2)^2_\infty$. This, together with (5.31), gives

$$\sum_{n\geq 0}[R(0,2,n) - R(1,2,n)]q^n = \frac{1}{(q^2;q^2)_\infty}\left(\sum_{m\in\mathbb{Z}} q^{4m^2} - q\sum_{m\in\mathbb{Z}} q^{4m^2-4m}\right)$$

which is equivalent to

$$\sum_{n\geq 0}[R(0,2,2n) - R(1,2,2n)]q^{2n} = \frac{1}{(q^2;q^2)_\infty}\sum_{m\in\mathbb{Z}} q^{4m^2}$$

and

$$\sum_{n\geq 0}[R(0,2,2n+1) - R(1,2,2n+1)]q^{2n+1} = -\frac{q}{(q^2;q^2)_\infty}\sum_{m\in\mathbb{Z}} q^{4m^2-4m}$$

and it is clear that the above two equations may be written as

$$\sum_{n\geq 0}[R(0,2,2n) - R(1,2,2n)]q^n = \frac{1}{(q;q)_\infty}\sum_{m\in\mathbb{Z}} q^{2m^2}$$

and

$$\sum_{n\geq 0}[R(1,2,2n+1) - R(0,2,2n+1)]q^n = \frac{1}{(q;q)_\infty}\sum_{m\in\mathbb{Z}} q^{2m^2-2m}.$$

All the coefficients of both these q-series are positive, which proves theorem 1.

Theorem 2.

Again, there is only one thing to look at, namely $R(0,3,n) - R(1,3,n)$. Putting $z = \omega$, where $\omega = e^{\frac{2\pi i}{3}}$ in (5.22) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n = \frac{1}{(\omega q;q)_\infty(\omega^{-1}q;q)_\infty}$$

so, by (5.23) (here $k = 3$),

$$R[0,3] + \omega R[1,3] + \omega^2 R[2,3] = \frac{(q;q)_\infty}{(q^3;q^3)_\infty} \tag{5.32}$$

Now

$$(q;q)_\infty = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{2}},$$

which is identity (1.15) and is equivalent to (1.16), the latter having been proved by using the bijection of Franklin in section 2.1. The above identity may be written as

$$[q;q^3](q^3;q^3)_\infty = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{2}} \tag{5.33}$$

which is also an instance of the triple product identity, specifically $q \to q^3$ and then set $z = q$ in (5.30). The sum on the right, over all integers n can be split into three seperate sums, i.e. (5.33) may be written as

$$\begin{aligned}
&[q;q^3](q^3;q^3)_\infty \\
&= \sum_{n=-3m} (-1)^n q^{\frac{3n^2+n}{2}} + \sum_{n=3m-1} (-1)^n q^{\frac{3n^2+n}{2}} + \sum_{n=-3m+1} (-1)^n q^{\frac{3n^2+n}{2}} \\
&= \sum_{m\in\mathbb{Z}} (-1)^m q^{\frac{27m^2-3m}{2}} - q\sum_{m\in\mathbb{Z}} (-1)^m q^{\frac{27m^2-15m}{2}} - q^2 \sum_{m\in\mathbb{Z}} (-1)^m q^{\frac{27m^2-21m}{2}}
\end{aligned}$$

and putting $q \to q^{27}$ in (5.30), followed by the appropriate value of z ($z = q^{12}$, $z = q^6$ and $z = q^3$ for the first, second and third sums) gives

$$[q;q^3](q^3;q^3)_\infty = [q^{12};q^{27}](q^{27};q^{27})_\infty - q[q^6;q^{27}](q^{27};q^{27})_\infty - q^2[q^3;q^{27}](q^{27};q^{27})_\infty$$

and so

$$[q;q^3] = \frac{1}{[q^3,q^6,q^9;q^{27}]} - \frac{q}{[q^3,q^9,q^{12};q^{27}]} - \frac{q^2}{[q^6,q^9,q^{12};q^{27}]}$$

which, together with (5.32), implies that

$$R[0,3] + \omega R[1,3] + \omega^2 R[2,3] = \frac{1}{[q^3,q^6,q^9;q^{27}]} - \frac{q}{[q^3,q^9,q^{12};q^{27}]} - \frac{q^2}{[q^6,q^9,q^{12};q^{27}]}$$

but $\omega^2 = -1-\omega$ and $R[2,3] = R[1,3]$, so

$$\sum_{n\geq 0} [R(0,3,n) - R(1,3,n)]q^n = \frac{1}{[q^3,q^6,q^9;q^{27}]} - \frac{q}{[q^3,q^9,q^{12};q^{27}]} - \frac{q^2}{[q^6,q^9,q^{12};q^{27}]}$$

which implies the following three results, from which (5.25) follows,

$$\sum_{n\geq 0} [R(0,3,3n) - R(1,3,3n)]q^n = \frac{1}{[q,q^2,q^3;q^9]},$$

$$\sum_{n\geq 0} [R(1,3,3n+1) - R(0,3,3n+1)]q^n = \frac{1}{[q,q^3,q^4;q^9]},$$

$$\sum_{n\geq 0} [R(1,3,3n+2) - R(0,3,3n+2)]q^n = \frac{1}{[q^2,q^3,q^4;q^9]}.$$

Theorem 3.

The quintuple product identity can be stated in the form,

$$[z;q][qz^{-2};q^2](q;q)_\infty = \sum_n z^{3n}q^{n(3n-1)/2}(1-zq^n). \tag{5.34}$$

A proof (of an equivalent version of (5.34)) can be found in 3.2 in [18].

The quintuple product identity will be used to find the 2-dissection of the Euler product, which is needed when looking at the behaviour of the birank mod 4.

Starting with (1.15),

$$(q;q)_\infty = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{3n^2+n}{2}}$$

$$= \sum_{n=-4m}(-1)^n q^{\frac{3n^2+n}{2}} + \sum_{n=4m+1}(-1)^n q^{\frac{3n^2+n}{2}} + \sum_{n=-4m-1}(-1)^n q^{\frac{3n^2+n}{2}} + \sum_{n=4m+2}(-1)^n q^{\frac{3n^2+n}{2}}$$

$$= \sum_{m\in\mathbb{Z}} q^{24m^2-2m} - \sum_{m\in\mathbb{Z}} q^{24m^2+14m+2} - \sum_{m\in\mathbb{Z}} q^{24m^2+10m+1} + \sum_{m\in\mathbb{Z}} q^{24m^2+26m+7}.$$

$$= \sum_{m\in\mathbb{Z}}(q^2)^{3m}(q^{16})^{\frac{m(3m-1)}{2}}(1-(q^2)(q^{16})^m) - q\sum_{m\in\mathbb{Z}}(q^6)^{3m}(q^{16})^{\frac{m(3m-1)}{2}}(1-(q^6)(q^{16})^m)$$

and so, putting $q \to q^{16}$ and $z = q^2$ and then $z = q^6$ in (5.34), gives

$$(q;q)_\infty = [q^2;q^{16}][q^{12};q^{32}](q^{16};q^{16})_\infty - q[q^6;q^{16}][q^4;q^{32}](q^{16};q^{16})_\infty. \tag{5.35}$$

Now, putting $z = i$ in (5.22) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)i^m q^n = \frac{1}{(iq;q)_\infty(i^{-1}q;q)_\infty}$$

$$= \frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty}$$

but,

$$\sum_m\sum_{n\geq 0} R(m,n)i^m q^n = R[0,4] + iR[1,4] - R[2,4] - iR[3,4]$$

$$= R[0,4] - R[2,4]$$

thus

$$\sum_{n\geq 0}[R(0,4,n) - R(2,4,n)]q^n = (q^2;q^4)_\infty. \tag{5.36}$$

Now $(q;q^2)_\infty$ is the generating function for partitions into distinct odd parts, those partitions having an odd number of parts being counted negatively. Such a partition of an even integer will have an even number of parts, and conversely a partition of an odd number will have an odd number of parts. Furthermore the only positive integer for which no such partition exists is 2. This proves that $n \neq 1 \Rightarrow R(0,4,4n) > R(2,4,4n)$ and that $R(0,4,4n) < R(2,4,4n)$. It also proves that $R(0,4,2n+1) = R(2,4,2n+1)$.

It has already been seen, by putting $z=-1$ into (5.22), that $R[0,2]-R[1,2]=(q;q^2)^2_\infty$. Since $R[0,2]=R[0,4]+R[2,4]$ and $R[1,2]=R[1,4]+R[3,4]=2R[1,4]$, it follows that

$$\sum_{n\geq 0}[R(0,4,n)+R(2,4,n)-2R(1,4,n)]q^n=(q;q^2)^2_\infty. \tag{5.37}$$

Adding (5.36) to (5.37) gives

$$\sum_{n\geq 0}[R(0,4,n)-R(1,4,n)]q^n=\frac{1}{2}\left((q^2;q^4)_\infty+(q;q^2)^2_\infty\right)$$

$$=\frac{(q;q^2)_\infty}{2}\left((-q;q^2)_\infty+(q;q^2)_\infty\right)\frac{(q^4;q^4)_\infty}{(q^4;q^4)_\infty}$$

$$=\frac{(q;q^2)_\infty}{2}\frac{[-q;q^4](q^4;q^4)_\infty+[q;q^4](q^4;q^4)_\infty}{(q^4;q^4)_\infty}.$$

Putting $q\to q^4$, and then $z=-q$ and $z=q$ into (5.30), it is seen that

$$R[0,4]-R[1,4]=\frac{1}{2}\frac{(q;q^2)_\infty}{(q^4;q^4)_\infty}\left(\sum_{n\in\mathbb{Z}}q^{2n^2-n}+\sum_{n\in\mathbb{Z}}(-1)^n q^{2n^2-n}\right)$$

and so

$$R[0,4]-R[1,4]=\frac{(q;q^2)_\infty}{(q^4;q^4)_\infty}\sum_{n\in\mathbb{Z}}q^{8n^2-2n} \tag{5.38}$$

which, together with $q\to q^{16}$ and $z=-q^6$ in (5.30), shows that

$$R[0,4]-R[1,4]=\frac{(q;q)_\infty[-q^6;q^{16}](q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^4;q^4)_\infty}$$

and so

$$R[0,4]-R[1,4]=\frac{(q;q)_\infty[q^{12};q^{32}](q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^4;q^4)_\infty[q^6;q^{16}]} \tag{5.39}$$

now the expression on the right is $(q;q)_\infty$ multiplied by a function of q^2, so with the 2-dissection of the Euler product (5.35), this becomes

$$R[0,4]-R[1,4]=$$

$$\frac{[q^2;q^{16}][q^{12};q^{32}]^2(q^{16};q^{16})^2_\infty}{(q^2;q^2)_\infty(q^4;q^4)_\infty[q^6;q^{16}]}-q\frac{[q^6;q^{16}][q^4;q^{32}][q^{12};q^{32}](q^{16};q^{16})^2_\infty}{(q^2;q^2)_\infty(q^4;q^4)_\infty[q^6;q^{16}]}. \tag{5.40}$$

so

$$R[0,4]-R[1,4]=\frac{1}{[q^6;q^{16}]^2[q^4,q^8;q^{32}]^2}-q\frac{(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^8;q^8)_\infty}.$$

This proves identities (5.42) and (5.43) below.

For the two other identities, the process is very similar. Subtracting (5.37) from (5.36), instead of adding them together, gives

$$\sum_{n\geq 0}[R(1,4,n)-R(2,4,n)]q^n = \frac{1}{2}\Big((q^2;q^4)_\infty - (q;q^2)^2_\infty\Big)$$

$$= \frac{(q;q^2)_\infty}{2}\Big((-q;q^2)_\infty - (q;q^2)_\infty\Big)\frac{(q^4;q^4)_\infty}{(q^4;q^4)_\infty}$$

$$= \frac{(q;q^2)_\infty}{2}\frac{[-q;q^4](q^4;q^4)_\infty - [q;q^4](q^4;q^4)_\infty}{(q^4;q^4)_\infty}.$$

and so, similarly to (5.38), this gives

$$R[1,4]-R[2,4] = q\frac{(q;q^2)_\infty}{(q^4;q^4)_\infty}\sum_{n\in\mathbb{Z}} q^{8n^2-6n}$$

which, together with $q \to q^{16}$ and $z=-q^2$ in (5.30), shows that

$$R[1,4]-R[2,4] = q\frac{(q;q)_\infty[-q^2;q^{16}](q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^4;q^4)_\infty}. \tag{5.41}$$

Now all that needs to be done, as above, is replace $(q;q)_\infty$ by the right hand side of (5.35) and simplify. This results in

$$R[1,4]-R[2,4] = q\frac{(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^8;q^8)_\infty} - q^2\frac{1}{[q^2;q^{16}]^2[q^8,q^{12};q^{32}]^2}$$

from which (5.44) and (5.45) below follow.

$$\sum_{n\geq 0}[R(0,4,2n)-R(1,4,2n)]q^n = \frac{1}{[q^3;q^8]^2[q^2,q^4;q^{16}]}, \tag{5.42}$$

$$\sum_{n\geq 0}[R(1,4,2n+1)-R(0,4,2n+1)]q^n = \frac{(q^8;q^8)^2_\infty}{(q;q)_\infty(q^4;q^4)_\infty}, \tag{5.43}$$

$$\sum_{n\geq 0}[R(2,4,2n)-R(1,4,2n)]q^n = q\frac{1}{[q;q^8]^2[q^4,q^6;q^{16}]}, \tag{5.44}$$

$$\sum_{n\geq 0}[R(1,4,2n+1)-R(2,4,2n+1)]q^n = \frac{(q^8,q^8)^2}{(q;q)_\infty(q^4;q^4)_\infty}. \tag{5.45}$$

This concludes the proof of theorem 3 (5.26), and also shows directly that $R(0,4,2n+1)=R(2,4,2n+1)$.

Theorem 4.

Putting $\omega = e^{2\pi i/5}$ in (5.22) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n = \frac{1}{(\omega q;q)_\infty(\omega^{-1}q;q)_\infty}$$

$$= \frac{(\omega^2 q;q)_\infty(\omega^{-2}q;q)_\infty(q;q)_\infty}{(q^5;q^5)_\infty}$$

the term on the right, by $q \to q^5$ and $z = \omega q$ in the other version of the triple product identity, namely (1.22), is equal to

$$\frac{1}{(q^5;q^5)_\infty}(1+(\omega+\omega^{-1})q-(\omega+\omega^{-1})q^6-q^{10}...)$$

$$= \frac{1}{(q^5;q^5)_\infty}\left(\sum_{n\in\mathbb{Z}}(-1)^n q^{25n^2+10n}+(\omega+\omega^{-1})q\sum_{n\in\mathbb{Z}}(-1)^n q^{25n^2+5n}\right)$$

which by suitable values in (1.22) is equal to

$$\frac{(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty}\left([q^{10};q^{25}]+(\omega+\omega^{-1})q[q^5;q^{25}]\right) = \frac{1}{[q^5;q^{25}]}+(\omega+\omega^{-1})\frac{q}{[q^{10};q^{25}]}$$

but,

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n = R[0,5]+\omega R[1,5]+\omega^2 R[2,5]+\omega^3 R[3,5]+\omega^4 R[4,5].$$

Thus (since $\omega^2 = -1-\omega-\omega^3-\omega^4$, and $R[1,5]=R[4,5]$ and $R[2,5]=R[3,5]$),

$$R[0,5]-R[2,5]+(\omega+\omega^4)(R[1,5]-R[2,5]) = \frac{1}{[q^5;q^{25}]}+(\omega+\omega^{-1})\frac{q}{[q^{10};q^{25}]}$$

This (and the irrationality of $\omega+\omega^{-1}$) proves (5.46) and (5.47), and so proves (5.27),

$$\sum_{n\geq 0}[R(0,5,n)-R(2,5,n)]q^n = \frac{1}{[q^5;q^{25}]}, \tag{5.46}$$

$$\sum_{n\geq 0}[R(1,5,n)-R(2,5,n)]q^n = \frac{q}{[q^{10};q^{25}]}. \tag{5.47}$$

Theorem 5.

The proof uses the well known expansion for $(q;q)_\infty$, i.e. Euler's Pentagonal Number Theorem. Also needed is that $q \to q^3$ and then $z = -q$ in (5.30) gives

$$\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^2}{(q;q)_\infty(q^6;q^6)_\infty} = \sum_{n\in\mathbb{Z}} q^{n(3n-1)/2} \tag{5.48}$$

and that $q \to q^2$ and then $z = -q$ in (5.30) gives

$$\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} \tag{5.49}$$

and finally $q \to q^6$ and then $z = -q$ in (5.34) gives

$$\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty}{(q;q)_\infty(q^6;q^6)_\infty} = \sum_{n\in\mathbb{Z}}(-1)^n q^{9n^2}(1+q^{6n+1}). \tag{5.50}$$

Now, putting $\omega = e^{2\pi i/6}$ in (5.22) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n = \frac{1}{(\omega q;q)_\infty(\omega^{-1}q;q)_\infty} = \frac{(q^2;q^2)_\infty(q^3;q^3)_\infty}{(q;q)_\infty(q^6;q^6)_\infty}$$

but

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n = R[0,6]+\omega R[1,6]+\omega^2 R[2,6]+\omega^3 R[3,6]+\omega^4 R[4,6]+\omega^5 R[5,6]$$

$$= R[0,6]+R[1,6]-R[2,6]-R[3,6]$$

$$\Rightarrow R[0,6]+R[1,6]-R[2,6]-R[3,6] = \frac{(q^2;q^2)_\infty(q^3;q^3)_\infty}{(q;q)_\infty(q^6;q^6)_\infty}. \tag{5.51}$$

Putting $\omega = e^{2\pi i/3}$ in (5.22) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n = \frac{1}{(\omega q;q)_\infty(\omega^{-1}q;q)_\infty} = \frac{(q;q)_\infty}{(q^3;q^3)_\infty}$$

but

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n = R[0,6]+\omega R[1,6]+\omega^2 R[2,6]+\omega^3 R[3,6]+\omega^4 R[4,6]+\omega^5 R[5,6]$$

$$= R[0,6]-R[1,6]-R[2,6]+R[3,6]$$

$$\Rightarrow R[0,6]-R[1,6]-R[2,6]+R[3,6] = \frac{(q;q)_\infty}{(q^3;q^3)_\infty}. \tag{5.52}$$

Also

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)(-1)^m q^n = \frac{1}{(-q;q)_\infty^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^2}$$

$$\Rightarrow R[0,6]-2R[1,6]+2R[2,6]-R[3,6] = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^2}. \tag{5.53}$$

The identity

$$\sum_{n\geq 0}[R(0,6,n)-R(2,6,n)]q^n = \frac{\sum_{n\in\mathbb{Z}} q^{6n^2+n}}{(q^3;q^3)_\infty}$$

follows from multiplying both equations (5.51) and (5.52) by $(q^3;q^3)_\infty$, adding the sum, using (5.48) (and the pentagonal number theorem), and dividing by $2(q^3;q^3)_\infty$. This proves the first part of (5.28), the rest comes from

$$\sum_{n\geq 0}[R(0,6,n)-R(3,6,n)]q^n = \frac{\sum_n(-1)^n q^{9n^2}}{(q^2;q^2)_\infty}$$

which follows in a similar way from (5.51) and (5.53) and then (5.49) and (5.50).

Theorem 6.

Putting $\omega = e^{2\pi i/8}$ in (5.22) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n = \frac{1}{(\omega q;q)_\infty(\omega^{-1}q;q)_\infty} = \frac{(q^4;q^8)_\infty}{(q;q)_\infty}\Big((\omega^3 q;q)_\infty(\omega^{-3}q;q)_\infty(q;q)_\infty\Big)$$

which, by (1.22) is equal to

$$\frac{(q^4;q^8)_\infty}{(q;q)_\infty}\Big(1+(\sqrt{2}-1)q-(\sqrt{2}-1)q^3-q^6-q^{10}+...\Big)$$
$$=\frac{(q^4;q^8)_\infty}{(q;q)_\infty}\Big(\sum_{n\geq 0}\alpha(n)q^{n(n+1)/2}+\sqrt{2}\sum_{n\geq 0}\beta(n)q^{n(n+1)/2}\Big)$$

where $\alpha(n) = 1$ if $n \equiv 0,2,5,7 \bmod 8$ else $= -1$, and $\beta(n) = 1$ if $n \equiv 1,6 \bmod 8$, $= -1$ if $n \equiv 2,5 \bmod 8$, else $= 0$. But, by $q \to q^2$ and $z = q$ in the triple product identity this becomes

$$\frac{(q^4;q^8)_\infty}{(q;q)_\infty}\Big(\frac{[q;q^8][q^6;q^{16}](q^8;q^8)^2_\infty}{[q^3;q^8](q^4;q^8)_\infty(q^{16};q^{16})_\infty}+q\sqrt{2}[q^2;q^{16}](q^{16};q^{16})_\infty\Big).$$

But

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R(m,n)\omega^m q^n$$

$$= R[0,8]+\omega R[1,8]+iR[2,8]+\omega^3R[3,8]-R[4,8]+\omega^5R[5,8]-iR[6,8]+\omega^7R[7,8]$$
$$= R[0,8]-R[4,8]+\sqrt{2}(R[1,8]-R[3,8]).$$

This implies

$$\sum_{n\geq 0}[R(0,8,n)-R(4,8,n)]q^n = \frac{(-q^4;q^8)_\infty}{[q^2;q^{16}][q^3;q^8]^2} \tag{5.54}$$

and

$$\sum_{n\geq 0}[R(1,8,n)-R(3,8,n)]q^n = \frac{q}{(q;q^2)_\infty(q^8;q^{16})_\infty[q^6;q^{16}]} \tag{5.55}$$

which together imply (5.29).

5.7 Related Identities

There is another approach to the birank, based on the following observation:

$$\sum_{\lambda \in \mathcal{P}} z^{\#(\lambda)} q^{wt(\lambda)} = \sum_{n \geq 0} \frac{z^n q^n}{(q;q)_n}. \tag{5.56}$$

This implies that, summing over all ordered pairs having birank congruent to r mod k,

$$\sum_{b(\pi) \equiv_k r} q^{wt(\pi)} = \sum_{s+t \equiv_k r} h(s,k)h(t,k) \tag{5.57}$$

where

$$h(s,k) := \sum_{n \geq 0} \frac{q^{kn+s}}{(q;q)_{kn+s}}.$$

This suggests that it may be worthwhile looking at the sum of $z^m/(q;q)_m$, summed over the (positive) values of m that satisfy a certain congruence. This in turn leads to the well known identity

$$\sum_{m \geq 0} \frac{z^m}{(q;q)_m} = \frac{1}{(z;q)_\infty}$$

which follows from $n \to \infty$ in (1.10). It follows from this that for n any natural number, and $\omega := e^{2\pi i/n}$,

$$\prod_j \left(\sum_{m \geq 0} \frac{w^{jm} z^m}{(q;q)_m} \right) = \prod_{d|n} (z^d; q^d)^{-\mu(n/d)} \tag{5.58}$$

where the product on the left is over all positive integers coprime to, and not greater than, n (and μ is the mobius function). For instance, if $n = 6$ then $\omega = e^{\frac{2pii}{6}}$ and (5.58) becomes

$$\left(1 + \frac{\omega z}{(q;q)_1} + \frac{\omega^2 z^2}{(q;q)_2} + \dots\right)\left(1 + \frac{\omega^5 z}{(q;q)_1} + \frac{\omega^{10} z^2}{(q;q)_2} + \dots\right) = \frac{(z^2;q^2)_\infty (z^3;q^3)_\infty}{(z;q)_\infty (z^6;q^6)_\infty}.$$

Now, consider the case $n = 3$. It follows from (5.56) that

$$\sum_{\pi \in \mathcal{P}^\epsilon} z^{b(\pi)} q^{wt(\pi)} = \left(\sum_{n \geq 0} \frac{z^n q^n}{(q;q)_n} \right) \left(\sum_{n \geq 0} \frac{z^{-n} q^n}{(q;q)_n} \right) \tag{5.59}$$

which is equal to

$$\left(h(0,3) + zh(1,3) + z^2 h(2,3)\right)\left(h(0,3) + z^{-1}h(1,3) + z^{-2}h(2,3)\right)$$

(this is true in general, but there's no point writing it like so unless $n = 3$).

Now, putting $z = q$ and expanding out gives the expression

$$h(0,3)^2 + h(1,3)^2 + h(2,3)^2 - h(0,3)h(1,3) - h(1,3)h(2,3) - h(2,3)h(0,3)$$

which by (5.58) is equal to $[q;q^3]$. This is equivalent to (5.32), it can be written as

$$\left(\sum_{n\geq 0}\frac{q^{3n}}{(q;q)_{3n}}\right)^2 + \left(\sum_{n\geq 0}\frac{q^{3n+1}}{(q;q)_{3n+1}}\right)^2 + \left(\sum_{n\geq 0}\frac{q^{3n+2}}{(q;q)_{3n+2}}\right)^2$$

$$-\left(\sum_{n\geq 0}\frac{q^{3n}}{(q;q)_{3n}}\right)\left(\sum_{n\geq 0}\frac{q^{3n+1}}{(q;q)_{3n+1}}\right) - \left(\sum_{n\geq 0}\frac{q^{3n+1}}{(q;q)_{3n+1}}\right)\left(\sum_{n\geq 0}\frac{q^{3n+2}}{(q;q)_{3n+2}}\right)$$

$$-\left(\sum_{n\geq 0}\frac{q^{3n+2}}{(q;q)_{3n+2}}\right)\left(\sum_{n\geq 0}\frac{q^{3n}}{(q;q)_{3n}}\right) = \frac{(q;q)_\infty}{(q^3;q^3)_\infty}.$$

Now for the case $n = 4$. Putting $z = q$ into (5.58) gives

$$\big(h(0,4)+ih(1,4)-h(2,4)-ih(3,4)\big)\big(h(0,4)+ih(1,4)-h(2,4)-ih(3,4)\big) = [q^2;q^4]$$

which is (5.36). It may be written as

$$h(0,4)^2 + h(1,4)^2 + h(2,4)^2 + h(3,4)^2 - 2h(0,4)h(2,4) - 2h(1,4)h(3,4)$$
$$= (q^2;q^4)_\infty,$$

or as

$$\left(\sum_{n\geq 0}(-1)^n\frac{q^{2n}}{(q;q)_{2n}}\right)^2 + \left(\sum_{n\geq 0}(-1)^n\frac{q^{2n+1}}{(q;q)_{2n+1}}\right)^2 = \frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty}.$$

So (5.58) can be used to help understand the birank, the above two identities are maybe not very interesting in themselves because (5.58) is probably not very interesting (but, as just explained, can be a way of tackling the birank).

Now for the interesting part: It follows from (5.40), together with (5.57), that

$$h(0,4)^2 + h(1,4)^2 + h(2,4)^2 + h(3,4)^2$$

$$-h(0,4)h(1,4) - h(1,4)h(2,4) - h(2,4)h(3,4) - h(3,4)h(0,4) =$$
$$\frac{(q;q^2)_\infty}{(q^4;q^4)_\infty}[-q^6;q^{16}](q^{16};q^{16})_\infty,$$

and from (5.41), together with (5.57), that

$$h(0,4)h(1,4) + h(1,4)h(2,4) + h(2,4)h(3,4) + h(3,4)h(0,4)$$

$$-2h(0,4)h(2,4) - 2h(1,4)h(3,4) = q\frac{(q;q^2)_\infty}{(q^4;q^4)_\infty}[-q^2;q^{16}](q^{16};q^{16})_\infty$$

and these two identities do not follow from (5.58).

5.8 Other Ranks

In this section, the 5 dissection of the Euler product is required. It is

$$(q;q)_\infty = \frac{[q^{10};q^{25}](q^{25};q^{25})_\infty}{[q^5;q^{25}]} - q(q^{25};q^{25})_\infty - q^2\frac{[q^5;q^{25}](q^{25};q^{25})_\infty}{[q^{10};q^{25}]} \tag{5.60}$$

which follows from lemma 6 in [5]. This chapter has dealt with the existence of a '2 dimensional rank', namely the birank. The aim here is to briefly describe a 4 dimensional rank, i.e. one defined on members of $\mathcal{P}^4$. This is not an unnatural avenue to persue, given the following easily proved congruence:

$$p_4(5n+3) \equiv p_4(5n+4) \equiv 0 \bmod 5. \tag{5.61}$$

So what 'super-rank' can be used to explain this congruence? One answer is: For $\kappa = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$ any ordered 4-tuple of partitions, define the super-rank $s(\kappa)$ as $s(\kappa) = 2\#\lambda(1) + \#\lambda(2) - \#\lambda(3) - 2\#\lambda(4)$. If $\omega = e^{2\pi i/5}$ then

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R^*(m,n)\omega^m q^n = \frac{1}{(\omega^2 q;q)(\omega q;q)(w^{-1}q;q)(\omega^{-2}q;q)} = \frac{(q;q)}{(q^5;q^5)}.$$

where $R^*(m,n)$ denotes the number of members of $\mathcal{P}^4$ having weight n and whose super-rank is m. So, by (5.60),

$$\sum_{m\in\mathbb{Z}}\sum_{n\geq 0} R^*(m,n)\omega^m q^n =$$

$$\frac{[q^{10};q^{25}](q^{25};q^{25})}{[q^5;q^{25}](q^5;q^5)} - q\frac{(q^{25};q^{25})}{(q^5;q^5)} - q^2\frac{[q^5;q^{25}](q^{25};q^{25})}{[q^{10};q^{25}](q^5;q^5)}.$$

But, for this super-rank,

$$R^*[0,5] - R^*[2,5] + (\omega+\omega^4)(R^*[1,5] - R^*[2,5]) =$$

$$\frac{[q^{10};q^{25}](q^{25};q^{25})}{[q^5;q^{25}](q^5;q^5)} - q\frac{(q^{25};q^{25})}{(q^5;q^5)} - q^2\frac{[q^5;q^{25}](q^{25};q^{25})}{[q^{10};q^{25}](q^5;q^5)}.$$

Hence, for all n, $R^*(1,5,n) = R^*(2,5,n)$ (which is not immediately obvious) and

$$\sum_{n\geq 0}[R^*(0,5,5n) - R^*(1,5,5n)]q^n = \frac{1}{[q;q^5]^2}$$

$$\sum_{n\geq 0}[R^*(1,5,5n+1) - R^*(0,5,5n+1)]q^n = \prod_{i\perp 5}\frac{1}{(1-q^i)}$$

(where the product is over all i coprime to 5)

$$\sum_{n\geq 0}[R^*(1,5,5n+2) - R^*(0,5,5n+2)]q^n = \frac{1}{[q^2;q^5]^2}$$

implying that $R^*(0,5,5n) > R^*(1,5,5n)$, $R^*(1,5,5n+1) > R^*(0,5,\ 5n+1)$ and $R^*(1,5,5n+2) > R^*(0,5,5n+2)$ except for $R^*(0,5,2) = R^*(1,5,2)$. The equality $R^*(0,5,5n+j) = R^*(1,5,5n+j)$ for $j=3$ or 4 also follows. This equality, together with $R^*(1,5,n) = R^*(2,5,n)$, implies (5.61).

5.9 Summary

It is worth noting that the only tools used in this paper have been the Jacobi Triple Product Identity, the Quintuple Product Identity and elementary results about algebraic independence ($\sqrt{2}$ is irrational, for example). This appears to be perhaps somewhat ironic, given that the super-ranks are a generalisation of Dyson's rank, and there is no equally elementary proof of his identities (see [5]). The appearance is deceptive. Whereas the Dyson rank involves both the first part and the number of parts of a partition, the super-rank of this paper is defined only in terms of the number of parts (of each partition in the ordered pair or k-tuple). An example of a proper generalisation would be: for an ordered pair of partitions, define a rank as "Dyson rank of first partition plus twice Dyson rank of second partition". Does this rank explain the congruence $p_{-2}(5n+2) \equiv 0 \bmod 5$? Just as there are other statistics similar to the rank, the crank for example, for ordinary partitions, there may well be other super ranks for ordered pairs (or triples or whatever) of partitions.

Finally, is there some combinatorial proof of these equalities and inequalities? Either a bijection that proves (say) $R(1,5,5n) = R(2,5,5n)$ or indeed some argument showing that $R(0,5,5n) > R(1,5,5n)$.

Chapter 6

A q-elliptic identity

6.1 The identity.

The aim of this chapter is to present a new approach to the following identity, which is stated as a

Theorem : Suppose that that $a_1, ..., a_N$ and $b_1, ..., b_N$ are nonzero complex numbers for which

$$a_1 a_2 ... a_N = b_1 b_2 ... b_N \tag{6.1}$$

and no two b_i are q-equivalent, which is to say that the ratio of two distinct b_is is not an integer power of q, i.e.

$$i \neq j \Rightarrow b_i \neq b_j q^t \tag{6.2}$$

for any integer t. When these conditions are satisfied the following identity holds;

$$\sum_{r=1}^{N} \frac{[a_1^{-1}b_r; q][a_2^{-1}b_r; q]...[a_N^{-1}b_r; q]}{[b_1^{-1}b_r; q][b_2^{-1}b_r; q]...[b_r \hat{b}_r^{-1}; q]...[b_r b_N^{-1}; q]} = 0 \tag{6.3}$$

(the $\hat{\ldots}$ indicates that the term $[b_r b_r^{-1}; q]$ is omitted). The standard proof (which can be found in [18]) of this involves analytic arguments, but my aim here is to present an involutive approach to this identity for the cases $N = 3$ and $N = 4$. For both these cases a bijective proof of an identity that follows from (6.3) and the Jacobi triple product identity is given. The triple product identity, which was proved in chapter 4, states

$$[z; q](q; q)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n z^n q^{\frac{n^2-n}{2}}. \tag{6.4}$$

Now, let $\pi = (\pi_1, \pi_2, ..., \pi_n)$ be an ordered n-tuple of integers, so π can be viewed as an element of $\mathbb{Z}^n$ or as a row matrix. The transpose of π will be written in bold, $\boldsymbol{\pi}$. In this chapter $\sigma(\pi)$ will be defined as $\sigma(\pi) := \pi_1 + \pi_2 + \ldots + \pi_n$, the sum of the parts and the weight of π is defined as

$$wt(\pi) := \frac{\pi_1^2 - \pi_1}{2} + \frac{\pi_2^2 - \pi_2}{2} + \ldots + \frac{\pi_n^2 - \pi_n}{2}.$$

6.2 The case $N = 3$.

The identity obtained from (6.3) by choosing particular values for (a_1, a_2, a_3) and (b_1, b_2, b_3) is the same as that which is obtained by chosing $(\lambda a_1, \lambda a_2, \lambda a_3)$ and $(\lambda b_1, \lambda b_2, \lambda b_3)$, where λ is any nonzero scalar. In particular, we can divide through by b_1, say. Having done this (6.1) ensures that when four of the five other entries are set, the last one is also determined. So if instead of (a_1, a_2, a_3) and (b_1, b_2, b_3) in the above theorem one picks (a_1, a_2, a_3) and $(1, b_1, b_2)$ then (6.1) ensures that $a_3 = b_1 b_2/(a_1 a_2)$. It follows that there is the following equivalent version of the theorem (in the case $N = 3$):

Whenever no two of $1, b_1$ and b_2 are q-equivalent, the following identity holds;

$$\frac{[a_1^{-1}; q][a_2^{-1}; q][a_1 a_2 b_1^{-1} b_2^{-1}; q]}{[b_1^{-1}; q][b_2^{-1}; q]} + \frac{[a_1^{-1} b_1; q][a_2^{-1} b_1; q][a_1 a_2 b_2^{-1}; q]}{[b_1; q][b_1 b_2^{-1}; q]}$$
$$+ \frac{[a_1^{-1} b_2; q][a_2^{-1} b_2; q][a_1 a_2 b_1^{-1}; q]}{[b_2; q][b_1^{-1} b_2; q]} = 0. \tag{6.5}$$

By using $[z^{-1}; q] = -z^{-1}[z; q]$, i.e. identity (1.3), this can be written as

$$b_1 b_2 \frac{[a_1^{-1}; q][a_2^{-1}; q][a_1 a_2 b_1^{-1} b_2^{-1}; q]}{[b_1; q][b_2; q]} - \frac{b_2}{b_1} \frac{[a_1^{-1} b_1; q][a_2^{-1} b_1; q][a_1 a_2 b_2^{-1}; q]}{[b_1; q][b_1^{-1} b_2; q]}$$
$$+ \frac{[a_1^{-1} b_2; q][a_2^{-1} b_2; q][a_1 a_2 b_1^{-1}; q]}{[b_2; q][b_1^{-1} b_2; q]} = 0 \tag{6.6}$$

and multiplication by $[b_1; q][b_2; q][b_1^{-1} b_2; q]$ gives

$$b_1 b_2 [a_1^{-1}; q][a_2^{-1}; q][a_1 a_2 b_1^{-1} b_2^{-1}; q][b_1^{-1} b_2; q]$$
$$- \frac{b_2}{b_1} [a_1^{-1} b_1; q][a_2^{-1} b_1; q][a_1 a_2 b_2^{-1}; q][b_2; q]$$
$$+ [a_1^{-1} b_2; q][a_2^{-1} b_2; q][a_1 a_2 b_1^{-1}; q][b_1; q] = 0. \tag{6.7}$$

Multiplying through by $(q; q)_\infty^4$ and using the Triple Product Identity (1.17) gives

$$b_1 b_2 \sum_{f \in \mathbb{Z}} (-1)^f a_1^{-f} q^{\frac{f^2-f}{2}} \sum_{g \in \mathbb{Z}} (-1)^g a_2^{-g} q^{\frac{g^2-g}{2}} \sum_{h \in \mathbb{Z}} (-1)^h a_1^h a_2^h b_1^{-h} b_2^{-h} q^{\frac{h^2-h}{2}} \sum_{k \in \mathbb{Z}} (-1)^k b_1^{-k} b_2^k q^{\frac{k^2-k}{2}}$$
$$- \frac{b_2}{b_1} \sum_{f \in \mathbb{Z}} (-1)^f a_1^{-f} b_1^f q^{\frac{f^2-f}{2}} \sum_{g \in \mathbb{Z}} (-1)^g a_2^{-g} b_1^g q^{\frac{g^2-g}{2}} \sum_{h \in \mathbb{Z}} (-1)^h a_1^h a_2^h b_2^{-h} q^{\frac{h^2-h}{2}} \sum_{k \in \mathbb{Z}} (-1)^k b_2^k q^{\frac{k^2-k}{2}}$$
$$+ \sum_{f \in \mathbb{Z}} (-1)^f a_1^{-f} b_2^f q^{\frac{f^2-f}{2}} \sum_{g \in \mathbb{Z}} (-1)^g a_2^{-g} b_2^g q^{\frac{g^2-g}{2}} \sum_{h \in \mathbb{Z}} (-1)^h a_1^h a_2^h b_1^{-h} q^{\frac{h^2-h}{2}} \sum_{k \in \mathbb{Z}} (-1)^k b_1^k q^{\frac{k^2-k}{2}} = 0. \tag{6.8}$$

This in turn can be written as an expression involving three sums, each of which is over all elements of $\mathbb{Z}^4$. To do this, let W_1, W_2 and W_3 be sets of ordered 4-tuples (so each W_i is the set $\mathbb{Z}^4$). Thus (6.8) becomes (with, for $\pi \in W_1, W_2$ or W_3, $\pi_1 = f$, $\pi_2 = g$, $\pi_3 = h$, $\pi_4 = k$),

$$\begin{aligned} &\sum_{\pi\in W_1} (-1)^{\sigma(\pi)} a_1^{\mu_1(\pi)} a_2^{\mu_2(\pi)} b_1^{\mu_3(\pi)} b_2^{\mu_4(\pi)} q^{wt(\pi)} \\ &- \sum_{\pi\in W_2} (-1)^{\sigma(\pi)} a_1^{\nu_1(\pi)} a_2^{\nu_2(\pi)} b_1^{\nu_3(\pi)} b_2^{\nu_4(\pi)} q^{wt(\pi)} \\ &+ \sum_{\pi\in W_3} (-1)^{\sigma(\pi)} a_1^{\tau_1(\pi)} a_2^{\tau_2(\pi)} b_1^{\tau_3(\pi)} b_2^{\tau_4(\pi)} q^{wt(\pi)} = 0 \end{aligned} \tag{6.9}$$

where

$$\mu_1(\pi) = -f + h, \quad \mu_2(\pi) = -g + h, \quad \mu_3(\pi) = -h - k + 1, \quad \mu_4(\pi) = -h + k + 1,$$

$$\nu_1(\pi) = -f + h, \quad \nu_2(\pi) = -g + p, \quad \nu_3(\pi) = f + g - 1, \quad \nu_4(\pi) = -h + k + 1,$$

$$\tau_1(\pi) = -f + h, \quad \tau_2(\pi) = -g + h, \quad \tau_3(\pi) = -h + k, \quad \tau_4(\pi) = f + g.$$

What is required is a means by which an element of one of the three sets can be paired off with a particular element of one of the other sets in a 'nice' way. What this means precisely will become clear later, but first it is helpful to decompose each of the three sets by defining $W_i^x := \{\pi \in W_i : \sigma(\pi) \equiv x \bmod 2\}$ ($x = 0$ or 1). It is now possible to define six maps (three pairs of maps),

$$\begin{aligned} \chi_1^2 &: W_1^0 \to W_2^0, & \chi_2^1 &: W_2^0 \to W_1^0, \\ \chi_1^3 &: W_1^1 \to W_3^0, & \chi_3^1 &: W_3^0 \to W_1^1, \\ \chi_2^3 &: W_2^1 \to W_3^1, & \chi_3^2 &: W_3^1 \to W_2^1. \end{aligned} \tag{6.10}$$

Thus, the notation χ_i^j has been used to denote a map from a subset of W_i to a subset of W_j. In fact, for $\chi_i^j(\pi)$ to be defined it is necessary that $\pi \in W_i$ and that either $\sigma(\pi) \equiv 0 \bmod 2$ and $(i, j) \in \{(1,2), (2,1), (3,1)\}$ or $\sigma(\pi) \equiv 1 \bmod 2$ and $(i, j) \in \{(1,3), (2,3), (3,2)\}$. Also, $\chi_i^j(\pi)$ may be written as π' and similarly, the transpose of $\chi_i^j(\pi)$ may be written in bold, $\boldsymbol{\pi}'$.

The maps are defined by

$$\pi' := \mathbf{A}_{\mathbf{i}}^{\mathbf{j}}\pi + \mathbf{c}_{\mathbf{i}}^{\mathbf{j}} \tag{6.11}$$

where the $\mathbf{A}_{\mathbf{i}}^{\mathbf{j}}$'s are four by four matrices and the $\mathbf{c}_{\mathbf{i}}^{\mathbf{j}}$'s are column matrices. They are

$$\mathbf{A}_{\mathbf{1}}^{\mathbf{2}} = \mathbf{A}_{\mathbf{2}}^{\mathbf{1}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad \mathbf{c}_{\mathbf{1}}^{\mathbf{2}} = \mathbf{c}_{\mathbf{2}}^{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{A}_{\mathbf{1}}^{\mathbf{3}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \quad \mathbf{c}_{\mathbf{1}}^{\mathbf{3}} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix},$$

$$\mathbf{A}_{\mathbf{3}}^{\mathbf{1}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \mathbf{c}_{\mathbf{3}}^{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{A}_{\mathbf{2}}^{\mathbf{3}} = \mathbf{A}_{\mathbf{3}}^{\mathbf{2}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{c}_{\mathbf{2}}^{\mathbf{3}} = \mathbf{c}_{\mathbf{3}}^{\mathbf{2}} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix},$$

Thus, for example, $\chi_1^3((f,g,h,k)) = (\frac{1}{2}(f-g-h+k+1), \frac{1}{2}(-f+g-h+k+1), \frac{1}{2}(-f-g+h+k+1), \frac{1}{2}(-f-g-h-k+3))$.

As an example, take $\pi := (-3,7,10,2) \in W_1^0$. Now, $\pi' = \chi_1^2(\pi) = (-10,0,3,-5)$. What is nice is that $wt(\pi) = 6+21+45+1 = 73$ and $wt(\pi') = 55+0+3+15 = 73$, the map is weight preserving. Furthermore, $\mu_1(\pi) = -(-3)+10 = 13$ and $\nu_1(\pi') = -(-10)+3 = 13$. In fact $\nu_\tau(\pi') = \mu_\tau(\pi)$ for $\tau \in \{1,2,3,4\}$. The other nice thing is that $\chi_2^1(\chi_1^2(\pi)) = \chi_2^1(-10,0,3,-5) = \pi$, or $\pi'' = \pi$. What all this means is that $(-3,7,10,2) \in W_1^0$ and $(-10,0,3,-5) \in W_2^0$ can be paired off with each other. From (6.9), $(-3,7,10,2)$ contributes

$$(-1)^{\sigma(\pi)} a_1^{\mu_1(\pi)} a_2^{\mu_2(\pi)} b_1^{\mu_3(\pi)} b_2^{\mu_4(\pi)} q^{wt(\pi)} = +a_1^{13} a_2^3 b_1^{-11} b_2^{-7} q^{73}$$

in the first sum, whilst $(-10,0,3,-5)$ gives

$$(-1)^{\sigma(\pi')} a_1^{\nu_1(\pi')} a_2^{\nu_2(\pi')} b_1^{\nu_3(\pi')} b_2^{\nu_4(\pi')} q^{wt(\pi')} = +a_1^{13} a_2^3 b_1^{-11} b_2^{-7} q^{73}$$

in the second sum (and so, since the second sum in (6.9) is preceeded by a minus sign, the expressions cancel each other out).

There is nothing unusual in the choice of π in the above paragraph. It is always the case that

$$\pi \in W_1^0 \Rightarrow wt(\chi_1^2(\pi)) = wt(\pi).$$

This follows from the fact that

$$\begin{aligned}
&\frac{1}{2}\left(\left(\frac{1}{2}(f-g-h-k)+1\right)^2-\left(\frac{1}{2}(f-g-h-k)+1\right)\right)\\
&+\frac{1}{2}\left(\left(\frac{1}{2}(-f+g-h-k)+1\right)^2-\left(\frac{1}{2}(-f+g-h-k)+1\right)\right)\\
&+\frac{1}{2}\left(\left(\frac{1}{2}(-f-g+h-k)+1\right)^2-\left(\frac{1}{2}(-f-g+h-k)+1\right)\right)\\
&+\frac{1}{2}\left(\left(\frac{1}{2}(-f-g-h+k)+1\right)^2-\left(\frac{1}{2}(-f-g-h+k)+1\right)\right)\\
&=\frac{f^2-f}{2}+\frac{g^2-g}{2}+\frac{h^2-h}{2}+\frac{k^2-k}{2}.
\end{aligned}$$

Indeed, for any π, the map defined on π is weight preserving; $wt(\chi_i^j(\pi)) = wt(\pi)$. Furthermore, since

$$\mathbf{A}_\mathbf{j}^\mathbf{i}(\mathbf{A}_\mathbf{i}^\mathbf{j}\pi + \mathbf{c}_\mathbf{i}^\mathbf{j}) + \mathbf{c}_\mathbf{j}^\mathbf{i} = \pi$$

it follows that that $\chi_j^i(\chi_i^j(\pi)) = \pi$, or $\pi'' = \pi$ so the maps are involutions. Equally straightforward are (for $1 \leq D \leq 4$),

$$\nu_D(\chi_1^2(\pi)) = \mu_D(\pi), \quad \mu_D(\chi_2^1(\pi)) = \nu_D(\pi),$$

$$\tau_D(\chi_1^3(\pi)) = \mu_D(\pi), \quad \mu_D(\chi_3^1(\pi)) = \tau_D(\pi),$$

$$\tau_D(\chi_2^3(\pi)) = \nu_D(\pi), \quad \nu_D(\chi_3^2(\pi)) = \tau_D(\pi).$$

It remains to investigate the effect of the relevant map on the parity of the sum of the parts of a given 4-tuple. This has been dealt with implicity in (6.10). In stating, for example, that $\chi_1^2 : W_1^0 \to W_2^0$ and not just $\chi_1^2 : W_1^0 \to W_2$, it is implicit that $\sigma(\chi_1^2(\pi)) \equiv \sigma(\pi) \bmod 2$. In fact, it is easy to verify that the maps χ_1^2, χ_2^1, χ_2^3 and χ_3^2 preserve parity whereas χ_1^3 and χ_3^1 reverse parity. Or more succintly,

$$i \equiv j \bmod 2 \iff \sigma(\chi_i^j(\pi)) \not\equiv \sigma(\pi) \bmod 2. \tag{6.12}$$

This implies cancellation occurs, since the expression in (6.9) involves sums that are alternately preceeded by plus and minus signs.

6.3 The case $N = 4$.

The following notation will be used in the proof of the theorem when $N = 4$. For $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7) \in \mathbb{Z}^7$, define

$$\delta_1(\pi) := 2\pi_5 - 2\pi_6 + 2\pi_7 - 1,$$

$$\delta_2(\pi) := -\pi_1 - \pi_2 - \pi_3 - \pi_4 + \pi_6 + \pi_7 + 1,$$

$$\delta_3(\pi) := -\pi_1 - \pi_2 - \pi_3 - \pi_4 + \pi_5 - \pi_7 + 2,$$

$$\delta_4(\pi) := -\pi_1 - \pi_2 - \pi_3 - \pi_4 - \pi_5 - \pi_6 + 3.$$

Let $\#(\pi) := \#\{y \in \{1,2,3,4\} : \delta_y(\pi) \equiv 0 \bmod 3\}$. It can be shown that for any $\pi \in \mathbb{Z}^7$, either $\#(\pi) = 4$ or $\#(\pi) = 1$. Without specifying at this stage what the subscript i is, define U_i^1 to be the set $\{\pi \in \mathbb{Z}^7 : \delta_1(\pi) \equiv 0 \bmod 3\}$. For $y = 2, 3$ or 4, define U_i^y to be the set $\{\pi \in \mathbb{Z}^7 : \delta_y(\pi) \equiv 0 \bmod 3, \delta_1(\pi) \not\equiv 0 \bmod 3\}$, the last condition ensuring that no π occurs in more than one of the four sets. Thus, for a given $\pi \in \mathbb{Z}^7$, either $\#(\pi) = 4$ and so $\pi \in U_i^1$ and none of the other three sets, or there is precisely one y such that $\delta_y(\pi) \equiv 0 \bmod 3$ (because $\#(\pi) \neq 4 \Rightarrow \#(\pi) = 1$) and this is the y for which $\pi \in U_i^y$. So any $\pi \in \mathbb{Z}$ is a member of precisely one U_i^y.

When $N = 4$, the theorem can be stated as: whenever no two of 1, b_1, b_2 and b_3 are q-equivalent the following identity holds;

$$\frac{[a_1^{-1};q][a_2^{-1};q][a_3^{-1};q][a_1a_2a_3b_1^{-1}b_2^{-1}b_3^{-1};q]}{[b_1^{-1};q][b_2^{-1};q][b_3^{-1};q]}$$
$$+\frac{[a_1^{-1}b_1;q][a_2^{-1}b_1;q][a_3^{-1}b_1;q][a_1a_2a_3b_2^{-1}b_3^{-1};q]}{[b_1;q][b_1b_2^{-1};q][b_1b_3^{-1};q]}$$
$$+\frac{[a_1^{-1}b_2;q][a_2^{-1}b_2;q][a_3^{-1}b_2;q][a_1a_2a_3b_1^{-1}b_3^{-1};q]}{[b_2;q][b_1^{-1}b_2;q][b_2b_3^{-1};q]}$$
$$+\frac{[a_1^{-1}b_3;q][a_2^{-1}b_3;q][a_3^{-1}b_3;q][a_1a_2a_3b_1^{-1}b_2^{-1};q]}{[b_3;q][b_1^{-1}b_3;q][b_2^{-1}b_3;q]} = 0. \tag{6.13}$$

which, using (1.3), can be rearranged to give

$$-b_1b_2b_3\frac{[a_1^{-1};q][a_2^{-1};q][a_3^{-1};q][a_1a_2a_3b_1^{-1}b_2^{-1}b_3^{-1};q]}{[b_1;q][b_2;q][b_3;q]}$$
$$+\frac{b_2b_3}{b_1^2}\frac{[a_1^{-1}b_1;q][a_2^{-1}b_1;q][a_3^{-1}b_1;q][a_1a_2a_3b_2^{-1}b_3^{-1};q]}{[b_1;q][b_1^{-1}b_2;q][b_1^{-1}b_3;q]}$$
$$-\frac{b_3}{b_2}\frac{[a_1^{-1}b_2;q][a_2^{-1}b_2;q][a_3^{-1}b_2;q][a_1a_2a_3b_1^{-1}b_3^{-1};q]}{[b_2;q][b_1^{-1}b_2;q][b_2^{-1}b_3;q]}$$
$$+\frac{[a_1^{-1}b_3;q][a_2^{-1}b_3;q][a_3^{-1}b_3;q][a_1a_2a_3b_1^{-1}b_2^{-1};q]}{[b_3;q][b_1^{-1}b_3;q][b_2^{-1}b_3;q]} = 0. \tag{6.14}$$

Multiplication by $[b_1;q][b_2;q][b_3;q][b_1^{-1}b_2;q][b_1^{-1}b_3;q][b_2^{-1}b_3;q]$ gives

$$\begin{aligned}
&-b_1b_2b_3[a_1^{-1};q][a_2^{-1};q][a_3^{-1};q][a_1a_2a_3b_1^{-1}b_2^{-1}b_3^{-1};q][b_1^{-1}b_2;q][b_1^{-1}b_3;q][b_2^{-1}b_3;q]\\
&\quad+\frac{b_2b_3}{b_1^2}[a_1^{-1}b_1;q][a_2^{-1}b_1;q][a_3^{-1}b_1;q][a_1a_2a_3b_2^{-1}b_3^{-1};q][b_2;q][b_3;q][b_2^{-1}b_3;q]\\
&\quad-\frac{b_3}{b_2}[a_1^{-1}b_2;q][a_2^{-1}b_2;q][a_3^{-1}b_2;q][a_1a_2a_3b_1^{-1}b_3^{-1};q][b_1;q][b_3;q][b_1^{-1}b_3;q]\\
&+[a_1^{-1}b_3;q][a_2^{-1}b_3;q][a_3^{-1}b_3;q][a_1a_2a_3b_1^{-1}b_2^{-1};q][b_1;q][b_2;q][b_1^{-1}b_2;q]=0. \qquad (6.15)
\end{aligned}$$

Multiplying through by $(q;q)_\infty^7$ and using the triple product identity (1.17) gives

$$\begin{aligned}
&-\sum_{\pi\in U_1}(-1)^{\sigma(\pi)}a_1^{\mu_1(\pi)}a_2^{\mu_2(\pi)}a_3^{\mu_3(\pi)}b_1^{\mu_4(\pi)}b_2^{\mu_5(\pi)}b_3^{\mu_6(\pi)}q^{wt(\pi)}\\
&+\sum_{\pi\in U_2}(-1)^{\sigma(\pi)}a_1^{\nu_1(\pi)}a_2^{\nu_2(\pi)}a_3^{\nu_3(\pi)}b_1^{\nu_4(\pi)}b_2^{\nu_5(\pi)}b_3^{\nu_6(\pi)}q^{wt(\pi)}\\
&-\sum_{\pi\in U_3}(-1)^{\sigma(\pi)}a_1^{\tau_1(\pi)}a_2^{\tau_2(\pi)}a_3^{\tau_3(\pi)}b_1^{\tau_4(\pi)}b_2^{\tau_5(\pi)}b_3^{\tau_6(\pi)}q^{wt(\pi)}\\
&+\sum_{\pi\in U_4}(-1)^{\sigma(\pi)}a_1^{\rho_1(\pi)}a_2^{\rho_2(\pi)}a_3^{\rho_3(\pi)}b_1^{\rho_4(\pi)}b_2^{\rho_5(\pi)}b_3^{\rho_6(\pi)}q^{wt(\pi)}=0
\end{aligned}$$

where U_1, U_2, U_3 and U_4 are four sets of ordered 7-tuples, $\pi=(k,l,m,n,p,r,s)\in\mathbb{Z}^7$, and

$$\mu_1(\pi)=-k+n,\quad \mu_2(\pi)=-l+n,\quad \mu_3(\pi)=-m+n,$$
$$\mu_4(\pi)=-n-p-r+1,\quad \mu_5(\pi)=-n+p-s+1,\quad \mu_6(\pi)=-n+r+s+1,$$
$$\nu_1(\pi)=-k+n,\quad \nu_2(\pi)=-l+n,\quad \nu_3(\pi)=-m+n,$$
$$\nu_4(\pi)=k+l+m-2,\quad \nu_5(\pi)=-n+p-s+1,\quad \nu_6(\pi)=-n+r+s+1,$$
$$\tau_1(\pi)=-k+n,\quad \tau_2(\pi)=-l+n,\quad \mu_3(\pi)=-m+n,$$
$$\tau_4(\pi)=-n+p-s,\quad \tau_5(\pi)=k+l+m-1,\quad \tau_6(\pi)=-n+r+s+1,$$
$$\rho_1(\pi)=-k+n,\quad \rho_2(\pi)=-l+n,\quad \rho_3(\pi)=-m+n,$$
$$\rho_4(\pi)=-n+p-s,\quad \rho_5(\pi)=-n+r+s,\quad \rho_6(\pi)=k+l+m.$$

Now, consider an element in any one of the four sets, $\pi\in U_i$. By earlier remarks there is a unique y such that $\pi\in U_i^y$. For this y define $\delta^y:=\delta_y(\pi)/3$.

There are sixteen maps,

$$\chi_i^i : U_i^1 \to U_i^1 \qquad (i = 1, 2, 3 \text{ or } 4),$$

$$\begin{aligned}
\chi_1^2 &: U_1^4 \to U_2^4, & \chi_2^1 &: U_2^4 \to U_1^4,\\
\chi_1^3 &: U_1^3 \to U_3^4, & \chi_3^1 &: U_3^4 \to U_1^3,\\
\chi_1^4 &: U_1^2 \to U_4^4, & \chi_4^1 &: U_4^4 \to U_1^2,\\
\chi_2^3 &: U_2^3 \to U_3^3, & \chi_3^2 &: U_3^3 \to U_2^3,\\
\chi_2^4 &: U_2^2 \to U_4^3, & \chi_4^2 &: U_4^3 \to U_2^2,\\
\chi_3^4 &: U_3^2 \to U_4^2, & \chi_4^3 &: U_4^2 \to U_3^2.
\end{aligned}$$

They are given by

$$\chi_i^i(\pi) = (k, l, m, n, p - \delta^1, r + \delta^1, s - \delta^1),$$

$$\begin{aligned}
\chi_1^2(\pi) = \chi_2^1(\pi) &= (k + \delta^4, l + \delta^4, m + \delta^4, n + \delta^4, p + \delta^4, r + \delta^4, s),\\
\chi_1^3(\pi) &= (k + \delta^3, l + \delta^3, m + \delta^3, n + \delta^3, -p + \delta^3 + 1, s + \delta^3, r),\\
\chi_3^1(\pi) &= (k + \delta^4, l + \delta^4, m + \delta^4, n + \delta^4, -p - \delta^4 + 1, s, r + \delta^4),\\
\chi_1^4(\pi) &= (k + \delta^2, l + \delta^2, m + \delta^2, n + \delta^2, -r + \delta^2 + 1, -s + \delta^2 + 1, p),\\
\chi_4^1(\pi) &= (k + \delta^4, l + \delta^4, m + \delta^4, n + \delta^4, s, -p - \delta^4 + 1, -r - \delta^4 + 1),\\
\chi_2^3(\pi) = \chi_3^2(\pi) &= (k + \delta^3, l + \delta^3, m + \delta^3, n + \delta^3, p - \delta^3, r, s + \delta^3),\\
\chi_2^4(\pi) &= (k + \delta^2, l + \delta^2, m + \delta^2, n + \delta^2, r - \delta^2, p, -s + \delta^2 + 1),\\
\chi_4^2(\pi) &= (k + \delta^3, l + \delta^3, m + \delta^3, n + \delta^3, r, p - \delta^3, -s - \delta^3 + 1),\\
\chi_3^4(\pi) = \chi_4^3(\pi) &= (k + \delta^2, l + \delta^2, m + \delta^2, n + \delta^2, p, r - \delta^2, s - \delta^2).
\end{aligned}$$

It is a straightforward task to show that the maps are weight preserving involutions and that

$$\begin{aligned}
\mu_D(\chi_1^1(\pi)) &= \mu_D(\pi), & \nu_D(\chi_2^2(\pi)) &= \nu_D(\pi),\\
\tau_D(\chi_3^3(\pi)) &= \tau_D(\pi), & \rho_D(\chi_4^4(\pi)) &= \tau_D(\pi),\\
\nu_D(\chi_1^2(\pi)) &= \mu_D(\pi), & \mu_D(\chi_2^1(\pi)) &= \nu_D(\pi),\\
\tau_D(\chi_1^3(\pi)) &= \mu_D(\pi), & \mu_D(\chi_3^1(\pi)) &= \tau_D(\pi),\\
\rho_D(\chi_1^4(\pi)) &= \mu_D(\pi), & \mu_D(\chi_4^1(\pi)) &= \rho_D(\pi),\\
\tau_D(\chi_2^3(\pi)) &= \nu_D(\pi), & \nu_D(\chi_3^2(\pi)) &= \tau_D(\pi),\\
\rho_D(\chi_2^4(\pi)) &= \nu_D(\pi), & \nu_D(\chi_4^2(\pi)) &= \rho_D(\pi),\\
\rho_D(\chi_3^4(\pi)) &= \tau_D(\pi), & \tau_D(\chi_4^3(\pi)) &= \rho_D(\pi).
\end{aligned}$$

It remains only to check that cancellation occurs. This is so because for any $(i, j) \in \{1, 2, 3, 4\}$ the statement (6.12) is true.

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