A: tables

A.0 From **2.2.3**, the cusps of $\Gamma_1(18)$ fall into sets :

table **A.0.0**

$$C_1^{(18)} = \left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\5\end{bmatrix}, \begin{bmatrix}0\\7\end{bmatrix}\right\}$$

$$C_2^{(18)} = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}1\\8\end{bmatrix}\right\}$$

$$C_3^{(18)} = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\15\end{bmatrix}\right\}$$

$$C_6^{(18)} = \left\{\begin{bmatrix}1\\6\end{bmatrix}, \begin{bmatrix}1\\12\end{bmatrix}\right\}$$

$$C_9^{(18)} = \left\{\begin{bmatrix}1\\9\end{bmatrix}, \begin{bmatrix}2\\9\end{bmatrix}, \begin{bmatrix}4\\9\end{bmatrix}\right\}$$

$$C_9^{(18)} = \left\{\begin{bmatrix}1\\18\end{bmatrix}, \begin{bmatrix}5\\18\end{bmatrix}, \begin{bmatrix}7\\18\end{bmatrix}\right\}$$

And I use **2.2.3** again to find the cusps of $\Gamma_1(32)$. They are:

table **A.0.1**

$$C_1^{(32)} = \left\{\begin{bmatrix}0\\k\end{bmatrix} : 1 \le k \le 15,\ k \text{ odd}\right\}$$

$$C_2^{(32)} = \left\{\begin{bmatrix}1\\2k\end{bmatrix} : 1 \le k \le 7,\ k \text{ odd}\right\}$$

$$C_4^{(32)} = \left\{\begin{bmatrix}1\\4k\end{bmatrix} : 1 \le k \le 7,\ k \text{ odd}\right\}$$

$$C_8^{(32)} = \left\{\begin{bmatrix}1\\8k\end{bmatrix}, \begin{bmatrix}3\\8k\end{bmatrix} : k = 1 \text{ or } 3\right\}$$

$$C_{16}^{(32)} = \left\{\begin{bmatrix}k\\16\end{bmatrix} : 1 \le k \le 7,\ k \text{ odd}\right\}$$

$$C_{32}^{(32)} = \left\{\begin{bmatrix}k\\0\end{bmatrix} : 1 \le k \le 15,\ k \text{ odd}\right\}$$

The following table gives orders of the various $s_9(k)$, $t_9(k)$ and $\eta(1)$, $\eta(3)$ and $\eta(9)$ at the cusps of $\Gamma_1(18)$. I also give lower bounds for the orders of the $\mathcal{P}_k$, the $\mathcal{E}_k$ and the $\mathcal{B}(m,n)$ of §**3.1** and of

$$\mathcal{S}(i,j,k) := q^{(24k-1)/72} \times \text{sum of theta products appearing in } S(i,j,k),$$

where $S(i,j,k)$ is given in §**3.4**.

table **A.0.2**

	C_1	C_2	C_3	C_6	1/9	2/9	4/9	1/18	7/18	5/18
s(1)	1/4	1/8	1/12	1/24	49/36	25/36	1/36	49/72	25/72	1/72
s(2)	1/4	1/8	1/12	1/24	25/36	1/36	49/36	25/72	1/72	49/72
s(3)	1/4	1/8	3/4	3/8	1/4	1/4	1/4	1/8	1/8	1/8
s(4)	1/4	1/8	1/12	1/24	1/36	49/36	25/36	1/72	49/72	25/72
t(0)	0	1/8	0	3/8	0	0	0	9/8	9/8	9/8
t(1)	0	1/8	1/3	1/24	1/9	4/9	16/9	49/72	25/72	1/72
t(2)	0	1/8	1/3	1/24	4/9	16/9	1/9	25/72	1/72	49/72
t(3)	0	1/8	0	3/8	1	1	1	1/8	1/8	1/8
t(4)	0	1/8	1/3	1/24	16/9	1/9	4/9	1/72	49/72	25/72
η(1)	3/4	3/8	1/4	1/8	1/12			1/24		
η(3)	1/4	1/8	3/4	3/8	1/4			1/8		
η(9)	1/12	1/24	1/4	1/8	3/4			3/8		
$\mathcal{P}_k$	≥ - 9/4	≥ - 9/8	≥ - 1/12	≥ - 1/24	≥ - 1/36			≥ - 1/72		
$\mathcal{B}$(m , n)	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0			≥ - 2/3		
$\mathcal{E}_k$	≥ - 1/4	≥ - 1/8	≥ - 1/12	≥ - 1/24	≥ - 1/36			≥ - 1/72		
$\mathcal{S}$(i , j , k)	≥ - 1/4	≥ - 1/8	≥ - 1/12	≥ - 1/24	≥ - 1/36			≥ -49/72		

And the next table does the same for the various functions appearing in §3.2 and for the $\mathcal{T}(i, j, k)$ (defined in the same way as the $\mathcal{S}(i, j, k)$).

table **A.0.3**

	C_1	C_2	C_4	1/8 1/24	3/8 3/24	1/16	3/16	5/16	7/16	1/32	3/32	5/32	7/32	9/32	11/32	13/32	15/32
s(2)	1/4	1/2	0	1/2	1/2	9/4	1/4	1/4	9/4	9/8	1/8	1/8	9/8	9/8	1/8	1/8	9/8
s(4)	1/4	1/2	1	0	0	1	1	1	1	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
s(6)	1/4	1/2	0	1/2	1/2	1/4	9/4	9/4	1/4	1/8	9/8	9/8	1/8	1/8	9/8	9/8	1/8
s(8)	1/4	1/2	1	2	2	0	0	0	0	0	0	0	0	0	0	0	0
t(1)	0	1/2	1/4	1/8	9/8	1/16	9/16	25/16	49/16	49/32	25/32	9/32	1/32	1/32	9/32	25/32	49/32
t(3)	0	1/2	1/4	9/8	1/8	9/16	49/16	1/16	25/16	25/32	1/32	49/32	9/32	9/32	49/32	1/32	25/32
t(5)	0	1/2	1/4	9/8	1/8	25/16	1/16	49/16	9/16	9/32	49/32	1/32	25/32	25/32	1/32	49/32	9/32
t(7)	0	1/2	1/4	1/8	9/8	49/16	25/16	9/16	1/16	1/32	9/32	25/32	49/32	49/32	25/32	9/32	1/32
η(1)	4/3	2/3	1/3	1/6		1/12				1/24							
η(4)	1/3	2/3	4/3	2/3		1/3				1/6							
η(16)	1/12	1/6	1/3	2/3		4/3				2/3							
$\mathcal{P}_k$	≥ - 16/3	≥ - 1/6	≥ - 1/12	≥ - 1/24		≥ - 1/48				≥ - 1/96							
$\mathcal{A}_*(m, n)$	≥ 0	≥ 0	≥ 0	≥ 0		≥ - 1/2				≥ - 1/4							
$\mathcal{T}(i, j, k)$	≥ - 4/3	≥ - 1/6	≥ - 1/12	≥ - 1/24		≥ - 25/48				≥ - 25/96							

A.1 In this section, I give the values of N(r, m, n) and M(r, m, n) needed to establish the theorems of §§ 3.1, 3.2 and 3.3.

table **A.1.0**

n	0	1	2	3	4
N(3, 9, 3n) = N(4, 9, 3n) = M(4, 9, 3n)	0	0	1	3	8
N(1, 9, 3n + 1)	0	1	1	5	11
N(2, 9, 3n + 1)	0	0	2	4	11
N(3, 9, 3n + 1)	0	1	2	5	12
N(4, 9, 3n + 1)	0	0	1	4	10
M(2, 9, 3n + 1)	0	1	2	5	12
M(3, 9, 3n + 1)	0	0	1	4	10
N(0, 9, 3n + 2) = N((4, 9, 3n + 2) = M(3, 9, 3n + 2)	0	1	2	6	15

table **A.1.1**

n	0	1	2	3	4
N(2, 8, 4n) = N(4, 8, 4n) = M(3, 8, 4n)	0	0	2	8	26
N(3, 8, 4n) - M(2, 8, 4n)	0	1	3	11	31
N(0, 8, 4n + 1)	1	1	6	15	41
N(1, 8, 4n + 1)	0	1	3	11	35
N(2, 8, 4n + 1)	0	1	4	14	39
N(3, 8, 4n + 1)	0	0	3	11	34
N(4, 8, 4n + 1)	0	2	4	14	40
M(0, 8, 4n + 1)	-1	1	2	11	33
M(1, 8, 4n + 1)	1	1	5	14	41
M(2, 8, 4n +1)	0	0	3	11	34
M(3, 8, 4n + 1)	0	2	5	15	40
N(0, 8, 4n + 2) = N((2, 8, 4n + 2) = M(1, 8, 4n + 2)	0	1	4	15	45
N(1, 8, 4n + 2) = M(2, 8, 4n + 2)	1	2	7	19	52
N(0, 8, 4n + 3)	1	3	8	24	66
N(1, 8, 4n + 3)	0	1	6	20	57
N(2, 8, 4n + 3)	1	3	9	25	66
N(3, 8, 4n + 3)	0	1	5	19	57
N(4, 8, 4n + 3)	0	2	8	24	64
M(1, 8, 4n + 3)	0	3	8	24	65
M(2, 8, 4n + 3)	0	1	6	20	57
M(3, 8, 4n +3)	1	2	8	24	65
M(4, 8, 4n + 3)	0	2	6	20	58

table **A.1.2**

n	0	1	2	3	4	5	6	7	8	9	10
N(2,12,2n) = N(5,12,2n)	0	0	0	1	2	3	6	11	18	32	52
N(1,12,2n+1) = N(4,12,2n+1	0	0	1	1	3	5	8	15	26	41	67

(Notice how the second row curiously resembles the Fibonacci series. This is, of course, illusory since the Fibonacci numbers increase as $\left((1 + \sqrt{5})/2\right)^n$, whereas $N(1, 12, 2n+1)$ is, presumably, more or less $p(2n+1)/12 \sim \exp\left(2\pi\sqrt{(n/3)}\right)/48n\sqrt{3}$ **[And1, 5.1.2]**)