1. Preparation

1.0 In this chapter, z, ζ, and q denote non-zero complex numbers, q being of modulus less than one.

$\sum_n$ denotes a sum over all integers n and $\sum_n'$ a sum over all non-zero integers n.

Define

$$(q)_\infty := (1-q)(1-q^2)(1-q^3)\ldots$$

$$[z\,;q]_\infty := (1-z)(1-zq)(1-zq^2)\ldots \times (1-z^{-1}q)(1-z^{-1}q^2)(1-z^{-1}q^3)\ldots$$

Each of these infinite products is absolutely convergent, since $|q| < 1$, and $[z\,;q]_\infty$ is a function of z that is meromorphic away from zero (i.e. meromorphic in every region $0 < r \le |z| \le r'$). Note that

$$[z^{-1}\,;q]_\infty = -z^{-1}[z\,;q]_\infty = [zq\,;q]_\infty \qquad \mathbf{1.0.0}$$

I shall make frequent use of the ***Jacobi triple product identity***, which states

$$[z\,;q]_\infty(q)_\infty = \sum_n (-)^n z^n q^{n(n-1)/2}. \qquad \mathbf{1.0.1}$$

There are numerous proofs of this illustrious identity. It was found by Jacobi **[Jac]** (though there is evidence that it was known to Gauss) in the course of his investigations into elliptic functions. Algebraic proofs have been given by Rademacher **[Rad, §100]**, Andrews **[And]** et al. and combinatorial proofs by Sylvester **[Syl]**, myself **[Lew]** et al.

Writing q^3 for q and q for z in **1.0.1** gives the *Pentagonal Number Theorem*, viz.

$$(q)_\infty = \sum_n (-)^n q^{n(3n-1)/2} \qquad \textbf{1.0.2}$$

1.0.2 was found by Euler **[Eul, §40]**, though it cannot be said that he gave a rigorous proof. A celebrated and most elegant combinatorial proof was given by Franklin **[Fra]**, **[And1, Thm 1.6]**, **[H+W, §19.11]**.

Another identity I shall use is the *quintuple product identity*, which was stated by Gordon **[Gor]** in the form :

$$\prod_{n=1}^{\infty}(1-s^n)(1-s^n t)(1-s^{n-1}t^{-1})(1-s^{2n-1}t^2)(1-s^{2n-1}t^{-2})$$
$$= \sum_n s^{(3n^2+n)/2}(t^{3n} - t^{-3n-1}), \quad |s| < 1,\ t \neq 0.$$

With the help of **1.0.1**, this identity may be recast in the form

$$\frac{[z^2;q]_\infty}{[-z;q]_\infty}(q)_\infty = \left([-z^3q;q^3]_\infty - z[-z^{-3}q;q^3]_\infty\right)(q^3)_\infty \qquad \textbf{1.0.3}$$

In fact, Gordon was anticipated by Watson who, 32 years earlier, found an equivalent identity **[Wat, pp 44-45]**. Apparently **[C+S]**, **1.0.3** may be traced to an elliptic sigma formula of Weierstrass. Following **[A+SD, lemma 5]**, I shall show how **1.0.3** is a simple consequence of **1.1.5** below.

1.1 A few definitions. Say that complex numbers z and z′ are *equivalent* (*q-equivalent* would be more precise) if $z' = zq^n$, for some integer n. Here, z, z′, q are as in **§1.0**. If f is a function meromorphic away from zero and $a \neq 0$, let $\operatorname{ord}(\mathfrak{f};a)$ denote the order of $\mathfrak{f}$ at a, i.e. that integer k such that

$$\lim_{z\to a}(z-a)^{-k}\mathfrak{f}(z)$$

is finite and not zero. If r is any positive real number, let

$$A_r := \{z : r|q| \leq |z| < r\},$$

i.e. the half-open annulus bounded by

$$C_r := \{z : |z| = r\}$$

and $C_{r|q|}$.

Suppose the function $\mathfrak{f}$ (meromorphic away from zero) satisfies

$$\mathfrak{f}(zq) = Cz^n\mathfrak{f}(z) \qquad \mathbf{1.1.0}$$

for every $z \neq 0$, where $C \neq 0$ and (integer) n are constant. Then it is plain that, if a and b are equivalent, $\operatorname{ord}(\mathfrak{f};a) = \operatorname{ord}(\mathfrak{f};b)$. So

$$N_{\mathfrak{f}} := \sum \operatorname{ord}(\mathfrak{f};x) = \sum_{y\in A_r}\operatorname{ord}(\mathfrak{f};y)$$

the first sum being over any complete set of inequivalent points x and, in the second sum, r is any positive real.

Lemma **[A+SD]**

Suppose $\mathfrak{f}$ is a complex-valued function on $\mathbb{C}$, meromorphic away from zero, and satisfying **1.1.0**. *If $\mathfrak{f}$ is not identically zero,*

$$N_{\mathfrak{f}} = -n. \qquad \mathbf{1.1.1}$$

(So, $\mathfrak{f}$ is either identically zero or has exactly n more inequivalent poles than inequivalent zeros)

Proof Choose r (as we may) so that $\mathfrak{f}$ has neither poles nor zeros on C_r. Then

$$N_{\mathfrak{f}} = \sum_{y \in A_r} \mathrm{ord}(\mathfrak{f}\,;y)$$
$$= \frac{1}{2\pi i}\int_{C_r} d\big(\log \mathfrak{f}(z)\big) - \frac{1}{2\pi i}\int_{C_{r|q|}} d\big(\log \mathfrak{f}(z)\big)$$

(the circles being traversed in an anticlockwise manner)

$$= \frac{1}{2\pi i}\int_{C_r} d\big(\log \mathfrak{f}(z)\big) - \frac{1}{2\pi i}\int_{C_r} d\big(\log \mathfrak{f}(zq)\big)$$
$$= -\frac{n}{2\pi i}\int_{C_r} d(\log z)$$

(by the hypothesis **1.1.0**)

$$= -n.$$ qed

As an example of the use of **1.1.1**, I'll give a proof of ***Winquist's identity*** **[Win]** This states:

$$\prod_{n=1}^{\infty}(1-q^n)^2(1-aq^{n-1})(1-a^{-1}q^n)(1-zq^{n-1})(1-z^{-1}q^n)$$
$$\times(1-azq^{n-1})(1-a^{-1}z^{-1}q^n)(1-a^{-1}zq^{n-1})(1-az^{-1}q^n)$$
$$= \sum_{i=0}^{\infty}\sum_{j}(-)^{i+j}\big\{(a^{-3i}-a^{3i+3})(z^{-3j}-z^{3j+1})$$
$$-(a^{-3j+1}-a^{3j+2})(z^{-3i-1}-z^{3j+2})\big\}q^{3i(i+1)/2+j(3j+1)/2} \qquad \mathbf{1.1.2}$$

As Hirschorn **[Hir]** observed, we can use **1.0.1** to rewrite **1.1.2** as

$$[z\,;q]_\infty[a\,;q]_\infty[za\,;q]_\infty[za^{-1};q]_\infty(q)_\infty^2 = \Big\{[z^3;q^3]_\infty\big([a^3q\,;q^3]_\infty - a[a^{-3}q\,;q^3]_\infty\big)$$
$$- za^{-1}[a^3;q^3]_\infty\big([z^3q\,;q^3]_\infty - z[z^{-3}q\,;q^3]_\infty\big)\Big\}(q^3)_\infty^2 \qquad \mathbf{1.1.3}$$

Let $\mathfrak{f}_L(z)$ and $\mathfrak{f}_R(z)$ denote the LHS and RHS of **1.1.3** and let $\mathfrak{f} = \mathfrak{f}_L - \mathfrak{f}_R$.

Then **1.0.0** shows that

$$f_L(zq) = -z^{-3}f_L(z), \quad f_R(zq) = -z^{-3}f_R(z)$$

and so f satisfies **1.1.0** with $C = -1$, $n = -3$. f plainly has no non-zero poles, so it is enough to show that f has four inequivalent zeros. Now

$$f_L(1) = 0 = f_R(1) \text{ and } f_L(a) = 0 = f_R(a)$$

and, if ω is one of the primitive third roots of 1,

$$\begin{aligned} f_L(\omega) &= -a^{-1}\omega[\omega\,;q]_\infty(q)_\infty[a\,;q]_\infty[\omega a\,;q]_\infty[\omega^2 a\,;q]_\infty(q)_\infty \\ &= -a^{-1}\omega(1-\omega)[a^3\,;q^3]_\infty(q^3)_\infty(q)_\infty \\ &= -a^{-1}\omega(1-\omega)[a^3\,;q^3]_\infty(q^3)_\infty[q\,;q^3]_\infty(q^3)_\infty \\ &= f_R(\omega). \end{aligned}$$

So f has four inequivalent zeros (unless a happens to be equivalent to a third root of 1, in which case it is easy to check that **1.1.3** holds).

Again suppose f is meromorphic away from zero, but now suppose f satisfies a stronger version of **1.1.0**, viz.

$$f(zq) = q^{-1}f(z). \qquad \textbf{1.1.4}$$

If a and b are equivalent points and f satisfies **1.1.4**, it is easy to see that f has the same residues at a and at b : $\mathrm{res}(f\,;a) = \mathrm{res}(f\,;b)$. The following lemma is really a well-known result about elliptic functions in disguise.

Lemma

Suppose $\mathfrak{f}$ is meromorphic away from zero and satisfies **1.1.4.** *Then*

$$\sum res(\mathfrak{f}\,;x) = 0, \qquad \textbf{1.1.5}$$

where the sum is over a complete set of inequivalent (non-zero) points x.

Proof Choose r so that $\mathfrak{f}$ has no poles on C_r. Then

$$\begin{aligned}\sum res(\mathfrak{f}\,;x) &= \sum_{x \in A_r} res(\mathfrak{f}\,;x)\\ &= \frac{1}{2\pi i}\int_{C_r}\mathfrak{f}(z)\,dz - \frac{1}{2\pi i}\int_{C_{r|q|}}\mathfrak{f}(z)\,dz\\ &= 0,\end{aligned}$$

because of **1.1.4.** qed

1.1.5 has the following elegant and useful

Corollary

Suppose that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are non-zero complex numbers, the b's being pairwise inequivalent, that satisfy

$$a_1a_2\ldots a_n = b_1b_2\ldots b_n. \qquad \textbf{1.1.6}$$

Then

$$\sum_{r=1}^{n} \frac{[a_1b_r^{-1};q]_\infty[a_2b_r^{-1};q]_\infty\ldots[a_nb_r^{-1};q]_\infty}{[b_1b_r^{-1};q]_\infty[b_2b_r^{-1};q]_\infty\ldots\hat{}\ldots[b_nb_r^{-1};q]_\infty} = 0, \qquad \textbf{1.1.7}$$

where ^ means that the term $[b_rb_r^{-1};q]$ is to be left out.

Proof Take

$$\mathfrak{f}(z) := \frac{[a_1 z\,;q]_\infty\,[a_2 z\,;q]_\infty \dots [a_n z\,;q]_\infty}{z\,[b_1 z\,;q]_\infty [b_2 z\,;q]_\infty \dots [b_n z\,;q]_\infty}\,(q)_\infty^2\,.$$

$\mathfrak{f}$ is meromorphic away from zero and the hypothesis **1.1.6**, with **1.0.0**, ensures that $\mathfrak{f}$ satisfies **1.1.4**. The b^{-1}'s make up a complete set of inequivalent poles of $\mathfrak{f}$ and the term displayed in **1.1.7** is $-\mathrm{res}(\mathfrak{f}\,;b_r^{-1})$. **1.1.7** now follows from **1.1.5**. ▢

1.1.7 appears (in a slightly different form) as an exercise in the book by Whittaker and Watson **[W+W, §20.53, ex. 3]**. It is a generalisation of **[A+SD, lemma 4]**.

The quintuple product identity **1.0.3** is a consequence of **1.1.7**. Write q^3 for q in **1.1.7** and take $n = 3$ and $(a_1, a_2, a_3\,; b_1, b_2, b_3) = (z^2, z^{-2}q^{-1}, q\,; 1, -z^{-1}q^{-1}, -zq)$. After some regrouping, out drops **1.0.3**.

1.2 Now for some applications of **1.1.1**.

For $k = 1$ or 3, define

$$T_k(z\,,\zeta,q) := \sum_n (-)^n \frac{\zeta^n q^{kn(n+1)/2}}{1 - zq^n}\,,$$

$$T_k^*(\zeta, q) := \sum_n{}' (-)^n \frac{\zeta^n q^{kn(n+1)/2}}{1 - q^n}\,.$$

Lemma

$$T_1(z\zeta,\zeta,q) = \frac{[\zeta^{-1};q]_\infty}{[-1;q]_\infty} T_1(-z,-1,q) + \frac{[-z\zeta;q]_\infty[-\zeta^{-1};q]_\infty(q)_\infty^2}{[z\zeta;q]_\infty[-z;q]_\infty[-1;q]_\infty} \qquad 1.2.0$$

$$\zeta T_1(z\zeta,\zeta,q) + T_1(z\zeta^{-1},\zeta^{-1},q) = \frac{[z;q]_\infty[\zeta^2;q]_\infty(q)_\infty^2}{[z\zeta;q]_\infty[\zeta;q]_\infty[z\zeta^{-1};q]_\infty} \qquad 1.2.1$$

$$\zeta^3 T_3(z\zeta,\zeta^3,q) + T_3(z\zeta^{-1},\zeta^{-3},q) = \zeta\frac{[\zeta^2;q]_\infty}{[\zeta;q]_\infty} T_3(z,1,q) + \frac{[\zeta;q]_\infty[\zeta^2;q]_\infty(q)_\infty^2}{[z\zeta;q]_\infty[z;q]_\infty[z\zeta^{1};q]_\infty} \qquad 1.2.2$$

Proof Write **1.2.0** as

$$\phi_1(z) = \phi_2(z) + \phi_3(z).$$

Then

$$\phi_1(zq) - \phi_2(zq) + z\big(\phi_1(z) - \phi_2(z)\big)$$

$$= \sum(-)^n\frac{\zeta^n q^{n(n+1)/2}}{1-z\zeta q^{n+1}} + z\sum(-)^n\frac{\zeta^n q^{n(n+1)/2}}{1-z\zeta q^n} - \frac{[\zeta^{-1};q]_\infty}{[-1;q]_\infty}\left(\sum\frac{q^{n(n+1)/2}}{1+zq^{n+1}} + z\sum\frac{q^{n(n+1)/2}}{1+zq^{n+1}}\right)$$

$$= \sum(-)^{n-1}\zeta^{n-1}q^{n(n-1)/2} - \frac{[\zeta^{-1};q]_\infty}{[-1;q]_\infty}\sum q^{n(n-1)/2}$$

$$= [\zeta^{-1};q]_\infty(q)_\infty - \frac{[\zeta^{-1};q]_\infty}{[-1;q]_\infty}[-1;q]_\infty(q)_\infty,$$

by **1.0.1**,

$$= 0.$$

Furthermore, **1.0.0** shows that $\mathfrak{f}_3(zq) = -\mathfrak{f}_3(z)$ and so $\mathfrak{f} := \mathfrak{f}_1 - \mathfrak{f}_2 - \mathfrak{f}_3$ satisfies **1.1.0** with $C = -1$ and $n = 1$. It follows from **1.1.1** that, to establish **1.2.0**, it is enough to show that $\mathfrak{f}$ is free of (non-zero) poles.

Now $\mathfrak{f}$ has poles at $z = \zeta^{-1}$, $z = -1$ and at equivalent points. We have

$$\mathrm{res}(\mathfrak{f}_1; -1) = 0, \ \mathrm{res}(\mathfrak{f}_2; -1) = \frac{[\zeta^{-1}; q]_\infty}{[-1; q]_\infty}, \ \mathrm{res}(\mathfrak{f}_3; -1) = -\frac{[\zeta^{-1}; q]_\infty}{[-1; q]_\infty}$$

and

$$\mathrm{res}(\mathfrak{f}_1; \zeta^{-1}) = -\zeta^{-1}, \quad \mathrm{res}(\mathfrak{f}_2; \zeta^{-1}) = 0, \ \mathrm{res}(\mathfrak{f}_3; -1) = -\zeta^{-1},$$

and so $\mathfrak{f}$ is indeed free from poles and **1.2.0** is proved. qed

Now **1.2.0** gives

$$\begin{aligned}
&\zeta T_1(z\zeta, \zeta, q) + T_1(z\zeta^{-1}, \zeta^{-1}, q) \\
&\qquad = \zeta \frac{[-z\zeta; q]_\infty [-\zeta^{-1}; q]_\infty (q)_\infty^2}{[z\zeta; q]_\infty [-z; q]_\infty [-1; q]_\infty} + \frac{[-z\zeta^{-1}; q]_\infty [-\zeta; q]_\infty (q)_\infty^2}{[z\zeta^{-1}; q]_\infty [-z; q]_\infty [-1; q]_\infty} \\
&\qquad = \frac{[z; q]_\infty [\zeta^2; q]_\infty (q)_\infty^2}{[z\zeta; q]_\infty [\zeta; q]_\infty [z\zeta^{-1}; q]_\infty}
\end{aligned}$$

by **1.1.6**, with $n = 3$ and $(a_1, a_2, a_3; b_1, b_2, b_3) = (-z, z^{-1}, 1; \zeta^{-1}, \zeta, -1)$. This is **1.2.1**

1.2.2 is **[ASD, 5.1]**. The proof given there is an enhanced version of the proof I've given of **1.2.0**. qed

I pause here to show how **0.3.8** may be derived from **1.2.0**. Setting $\zeta = 1$ in **1.2.0** gives

$$T_1(z,1,q) = \sum_n (-)^n \frac{q^{n(n+1)/2}}{1 - zq^n} = \frac{(q)_\infty^2}{[z;q]_\infty} \qquad 1.2.3$$

Now it is obvious from the definition of N_V that

$$\sum_{m,n} N_V(m,n) z^m q^n = \frac{(1-z)(q)_\infty}{[z;q]_\infty}$$

and, since $P(q)_\infty = 1$, **1.2.3** yields **0.3.8.**

Define

$$\mathcal{g}_M(z) = \mathcal{g}_M(z,q) := -zT_1(z^2,z,q) - T_1^*(z^{-1},q),$$

$$\mathcal{g}_N(z) = \mathcal{g}_N(z,q) := z\frac{[z^2;q]_\infty}{[z;q]_\infty} T_3(z,1,q) - z^3T_3(z^2,z^3,q) - T_3^*(z^{-3},q).$$

$$h(z) = h(z,q) := \frac{[z;q]_\infty}{[-1;q]_\infty} T_1(-z,-1,q) - T_1^*(z^{-1},q)$$

$$H(z) = H(z;q) := \frac{[z^2;q]_\infty^3 (q)_\infty^2}{[z;q]_\infty^3 [z^3;q]_\infty}$$

Note the following two properties of H(z)

$$H(zq) = H(z), \qquad 1.2.4$$

$$H(z) + H(z^{-1}) = 0. \qquad 1.2.5$$

Now the poles of

$$\mathcal{g}_M(z),\ \mathcal{g}_M(z^3),\ \mathcal{g}_N(z),\ \mathcal{g}_N(z^3),\ h(z),\ h(z^3),\ H(z),\ H(z^2) \text{ and } H(-z) \qquad 1.2.6$$

are all simple and they all lie in

$$\prod := \{z : z^6 = q^n,\ n \in \mathbf{Z}\}.$$

Break $\prod$ up into sets $\prod_1$, $\prod_2$, $\prod_3$ and $\prod_6$, where

$\prod_r := \{z : z^r = q^n,\ 0 < r \in \mathbb{Z},\ n \in \mathbb{Z},\ r \text{ minimal with this property}\}.$

Then the residues of the functions **1.2.6** at $a \in \prod$ are as in the following table :

residue at $a \in$	$\prod_1$	$\prod_2$	$\prod_3$	$\prod_6$
$g_M(z)$	$a/2$	$-a/2$	0	0
$g_M(z^3)$	$a/6$	$-a/6$	$a/6$	$-a/6$
$g_N(z)$	$-3a/2$	$-a/2$	0	0
$g_N(z^3)$	$-a/2$	$-a/6$	$-a/2$	$-a/6$
$h(z)$	0	$a/3$	0	$a/3$
$H(z)$	$-8a/3$	0	$a/3$	0
$H(-z)$	0	$8a/3$	0	$-a/3$
$H(z^2)$	$-4a/3$	$-4a/3$	$a/6$	$a/6$

1.2.7

Lemma

$$g_M(z) - g_M(zq) = -1, \quad \mathbf{1.2.8}$$

$$g_M(z) + g_M(z^{-1}) = 0, \quad \mathbf{1.2.9}$$

$$3g_M(z) - g_M(z^3) = H(z^2) - H(z), \quad \mathbf{1.2.10}$$

$$g_N(z) - g_N(zq) = -3, \quad \mathbf{1.2.11}$$

$$g_N(z) + g_N(z^{-1}) = -2, \quad \mathbf{1.2.12}$$

$$3g_N(z) - g_N(z^3) = H(z^2) + H(z) - 2, \quad \mathbf{1.2.13}$$

$$h(z) - h(zq) = -1, \quad \mathbf{1.2.14}$$

$$h(z) + h(z^{-1}) = 0, \quad \mathbf{1.2.15}$$

$$2h(z) - h(z^2) = \frac{[z;q]_\infty^2 [z^2;q]_\infty (q)_\infty^2}{[-z;q]_\infty^2 [-z^2;q]_\infty^2 [-1;q]_\infty} \quad \mathbf{1.2.16}$$

$$3h(z) - h(z^3) = H(-z). \quad \mathbf{1.2.17}$$

Proof Of these ten identities, **1.2.8**, **1.2.9**, **1.2.11**, **1.2.12**, **1.2.14** and **1.2.15** are a matter of elementary manipulations. (**1.2.11** and **1.2.12** are, respectively, **5.11** and **5.12** in **[A+SD]**}. If we denote by $\mathfrak{f}(z)$ the difference between the left and right hand sides of **1.2.10** (or **1.2.13** or **1.2.17**), **1.2.8** (or **1.2.11** or **1.2.14**) together with **1.2.4** shows that

$$\mathfrak{f}(zq) = \mathfrak{f}(z).$$

Now examination of the table **1.2.7** shows that each of these three functions $\mathfrak{f}$ has no (non-zero) poles. Furthermore, **1.2.9** (or **1.2.12** or **1.2.15**) with **1.2.5** shows that, in each case,

$$\mathfrak{f}(z) + \mathfrak{f}(z^{-1}) = 0$$

and so $\mathfrak{f}(1) = 0$. Now **1.2.10**, **1.2.13** and **1.2.17** follow from **1.1.1**.

The final identity, **1.2.16** may be proved in much the same way. ▭qed

1.2.10, **1.2.13** and **1.2.17** suggest that

$$g_N(z) = h(-z^2) + h(-z) - 1 \text{ and } g_M(z) = h(-z^2) - h(-z).$$

These are both true; as I do not need these facts I leave their proofs as easy exercises for the reader.

1.3 For integers r, m and (odd) k, set

$$S_k(r, m) := \sum_n{}' (-)^n \frac{q^{n(kn+1)/2 + rn}}{1 - q^{mn}}$$

By reversing the order of summation, we find that

$$S_k(m-1-r, m) = -S_k(r, m). \tag{1.3.0}$$

Furthermore,

$$\begin{aligned} S_k(r, m) - S_k(r+m, m) &= \sum_n{}' (-)^n q^{n(kn+1)/2+m} \\ &= [q^{r+(k+1)/2}; q]_\infty (q^k)_\infty - 1. \end{aligned} \tag{1.3.1}$$

by **1.0.1**. In fact, the only case I need of **1.3.1** is that with $k = 3$, $m = 8$ and $r = 2$, i.e.

$$S_3(2, 8) = S_3(10, 8) - q^{-1}P - 1. \tag{1.3.2}$$

Define numbers $N_k(m, n)$ by

$$\sum_{m,n} N_k(m, n) z^m q^n := (1-z)P \sum_n (-)^n \frac{q^{n(kn+1)/2}}{1 - zq^n} \tag{1.3.3}$$

and let $N_k(r, m, n) = \sum_t N_k(r+tm, n)$. So, as I noted at **0.2.4** and **0.3.8**, $N_3 = N$ and $N_1 = N_V = M$. Set $\omega := \exp(2\pi i/m)$. Then **1.3.3** gives

$$\begin{aligned} \sum_n N_k(r, m, n) q^n &= \frac{1}{m} \sum_{s=0}^{m-1} \omega^{-sr} \sum_{u,n} N_k(u, n) \omega^{su} q^n \\ &= \frac{1}{m} \sum_{s=0}^{m-1} \omega^{-sr} (1-\omega^s) P \sum_n{}' (-)^n \frac{q^{n(kn+1)/2}}{1 - \omega^s q^n} + \frac{1}{m} P \sum_{s=0} \omega^{-sr} \\ &= P \sum_n{}' (-)^n q^{n(kn+1)/2} \sum_{s=0}^{m-1} \frac{\omega^{-sr} - \omega^{-s(r-1)}}{1 - \omega^s q^n} \quad (+\ P, \text{ when } r = 0) \\ &= P\big(S_k(0, m) - S_k(m-1, m) + 1\big) \\ &= P\big(2S_k(0, m) + 1\big), \text{ when } r = 0, \text{ and} \\ &= P\big(S_k(r, m) - S_k(r-1, m)\big), \text{ when } 1 \le r < m, \end{aligned} \tag{1.3.4}$$

since it is easy to see that

$$\sum_{s=0}^{m-1}\omega^{-sr}(1 - \omega^s x)^{-1} = x^r(1 - x^m)^{-1}, \text{ for } 0 \le r < m.$$

From now on, I'll write S_M for S_1 and S_N for S_3.

1.4 Suppose $X = x_0 + x_1 q + x_2 q^2 + \ldots$ is a power series in the variable q. For positive integer m, define power series $X_r^{(m)}$ $(0 \le r < m)$ by

$$X_r^{(m)} := x_r q^r + x_{r+m} q^{r+m} + x_{r+2m} q^{r+2m} \ldots .$$

I call these $X_r^{(m)}$ the *(m-)components* of X.

I shall need to know suitable expressions for $P_r^{(m)}$ for m = 2, 3 and 4. Now Euler **[Eul]** remarked that

$$P := \sum_n p(n)q^n = (q)_\infty^{-1}.$$

Sorting the RHS of **1.0.2** into even and odd powers of q, we find with the help of **1.0.1** and **1.0.3** that

$$(q)_\infty = (q^{16})_\infty [q^2; q^8]_\infty \big([-q^6; q^{16}]_\infty - [-q^2; q^{16}]_\infty q\big) \qquad \textbf{1.4.0}$$

Since

$$(q)_\infty(-q)_\infty = (q^2)_\infty^2 [q^2; q^8]_\infty,$$

we have from **1.4.0**

$$\begin{aligned} P = (q)_\infty^{-1} &= (-q)_\infty (q^2)_\infty^{-2} [q^2; q^8]_\infty^{-1} \\ &= (q^{16})_\infty (q^2)_\infty^{-2} \big([-q^6; q^{16}]_\infty + [-q^2; q^{16}]_\infty q\big). \qquad \textbf{1.4.1} \end{aligned}$$

$\therefore$

$$P_0^{(2)} = \frac{(q^{16})}{(q^2)_\infty^2}[-q^6;q^{16}]_\infty \quad \text{and} \quad P_1^{(2)} = \frac{(q^{16})_\infty}{(q^2)_\infty^2}[-q^2;q^{16}]_\infty q \qquad \mathbf{1.4.2}$$

In like manner, we have

$$(q)_\infty = (q^{27})_\infty\left([q^{12};q^{27}] - [q^6;q^{27}]_\infty q - [q^3;q^{27}]_\infty q^2\right) \qquad \mathbf{1.4.3}$$

With $\omega := \exp(2\pi i/3)$, we have

$$(q)_\infty(\omega q)_\infty(\omega^2 q)_\infty = (q^3)_\infty^3[q^3;q^9]_\infty$$

and so

$$P = (\omega q)_\infty(\omega^2 q)_\infty(q^3)_\infty^{-3}[q^3;q^9]_\infty^{-1}$$

from which, by way of **1.4.3**, we find that

$$P_0^{(3)} = \frac{(q^{27})^2(q^9)_\infty}{(q^3)_\infty^4}\left([q^{12};q^{27}]_\infty^2 - [q^6;q^{27}]_\infty[q^3;q^{27}]_\infty q^3\right)$$

$$P_1^{(3)} = \frac{(q^{27})^2(q^9)_\infty}{(q^3)_\infty^4}\left([q^3;q^{27}]_\infty^2 q^3 + [q^{12};q^{27}]_\infty[q^6;q^{27}]_\infty\right)q$$

$$P_2^{(3)} = \frac{(q^{27})^2(q^9)_\infty}{(q^3)_\infty^4}\left([q^6;q^{27}]_\infty^2 + [q^3;q^{27}]_\infty[q^{12};q^{27}]_\infty\right)q^2 \qquad \mathbf{1.4.4}$$

1.4.2 and **1.4.4** were found by Kolberg **[Kol]**. The derivations I've given here are essentially the same as his.

Now to find similar expressions for the $P_r^{(4)}$. **1.0.1** gives

$$[-q^6;q^{16}]_\infty(q^{16})_\infty = \left([-q^{28};q^{64}]_\infty + [-q^4;q^{64}]_\infty q^6\right)(q^{64})_\infty$$

and

$$[-q^2;q^{16}]_\infty(q^{16})_\infty = \Big([-q^{20};q^{64}]_\infty + [-q^{12};q^{64}]_\infty q^2\Big)(q^{64})_\infty.$$

Substituting these expressions into **1.4.1** gives

$$P = (q^{32})_\infty^2(q^4)_\infty^{-4}\Big([-q^{12};q^{32}]_\infty + [-q^4;q^{32}]_\infty q^2\Big)^2$$
$$\times\Big([-q^{28};q^{64}]_\infty + [-q^{20};q^{64}]_\infty q + [-q^4;q^{64}]_\infty q^6 + [-q^{28};q^{64}]_\infty q^3\Big)(q^{64})_\infty$$

and we have

$$\begin{aligned}
P_0^{(4)} &= [-q^{28};q^{64}]_\infty K + [-q^4;q^{64}]_\infty L\, q^8,\\
P_1^{(4)} &= \Big([-q^{20};q^{64}]_\infty K + [-q^{12};q^{64}]_\infty L\, q^4\Big)q\,,\\
P_2^{(4)} &= \Big([-q^4;q^{64}]_\infty K\, q^4 + [-q^{28};q^{64}]_\infty L\Big)q^2,\\
P_3^{(4)} &= \Big([-q^{12};q^{64}]_\infty K + [-q^{20};q^{64}]_\infty L\Big)q^3,
\end{aligned} \qquad \mathbf{1.4.5}$$

where

$$\begin{aligned}
K &:= (q^{64})_\infty(q^{32})_\infty^2(q^4)_\infty^{-4}\Big([-q^{12};q^{32}]_\infty^2 + [-q^4;q^{32}]_\infty^2 q^8\Big)\\
&= (q^{64})_\infty(q^{32})_\infty^2(q^{16})_\infty(q^4)_\infty^{-5}[q^{16};q^{64}]_\infty^4 \qquad \text{(by } \mathbf{1.1.6}\text{)}
\end{aligned} \qquad \mathbf{1.4.6}$$

and

$$L := 2(q^{64})_\infty(q^{32})_\infty^2(q^4)_\infty^{-4}[-q^4;q^{16}]_\infty\,. \qquad \mathbf{1.4.7}$$

I note here three identities, each a consequence of **1.0.1**.

$$(q)_\infty = (q^{64})_\infty\sum_{k=-1}^{2}(-)^k q^{k(3k+1)/2}\,\frac{[q^{56-16k};q^{64}]_\infty}{[-q^{28-8k};q^{64}]_\infty} \qquad \mathbf{1.4.8}$$

(?? (something like this is true))

This may be proved by writing $n = 8m+r$ in **1.0.2** and using **1.0.1** and **1.0.3**.

The second of these identities is

$$\Big([q^{12-3u};q^{27}]_\infty - q^{1-u}[q^{21-3u};q^{27}]_\infty - q^{2+u}[q^{3-3u};q^{27}]_\infty\Big)(q^{27})_\infty$$
$$= [q^{u+2};q^3]_\infty(q^3)_\infty = \alpha_u(q)_\infty$$

where, writing $u = 3m + s$ with $s = 0$, 1 or -1,

$$\alpha_u = \begin{cases} (-)^m q^{m(1-s)-3m(m+1)/2}, & \text{if } s = 0 \text{ or } -1 \\ 0, & \text{if } s = 1. \end{cases} \tag{1.4.9}$$

The last of these three identities is

$$\Big([-q^{22-4u}; q^{48}]_\infty - q^{u+2}[-q^{10-4u}; q^{48}]_\infty + q^{2u+7}[-q^{-2-4u}; q^{48}]_\infty + q^{3u+15}[-q^{-14-4u}; q^{48}]_\infty\Big)(q^{48})_\infty = [q^{u+2}; q^3]_\infty (q^3)_\infty = \alpha_u (q)_\infty \tag{1.4.10}$$

with α_u as in **1.4.9**.

1.5 Set

$$N_m(r) := \sum N(r, m, n) q^n \quad \text{and} \quad M_m(r) := \sum M(r, m, n) q^n.$$

In this section, I find usable expressions for the $N_8(r)$ and the $M_8(r)$, in particular expressions in which their 4-components are apparent.

We have

$$S_N(3u + 1, 8)) = \sum_n{}' (-)^n \frac{q^{(3n+1)/2 + (3u+1)n}}{1 - q^{8n}}$$

which, on writing $n = 8m + r$, becomes

$$= \sum_{r \bmod 8} (-)^r q^{3r(r+2u+1)/2} \sum_m \frac{q^{3(8r + 8u - 28)m} q^{64.3m(m+1)/2}}{1 - q^{8r} q^{64m}}$$

(with $\sum_m{}'$ for $\sum_m$ when $r \equiv 0 \bmod 8$)

$$= \sum_{\substack{r=-u \\ r\neq 0}}^{3-u} (-)^r q^{3r(r+2u+1)/2}\Big(T_3(q^{8r}, -q^{3(8r+8u-28)}, q^{64}) - q^{3(28-8r-8u)}T_3(q^{56-8r-16u}, -q^{3(28-8r-8u)}, q^{64})\Big)$$

$$+ T_3^*(-q^{3(8u-28)}, q^{64}) - q^{3(28-8u)}T_3(q^{2(28-8u)}, -q^{3(28-8u)}, q^{64}) \qquad \mathbf{1.5.0}$$

For $m \not\equiv \pm n \bmod 16$, set

$$A_M(m,n) = A_M(m,n;q) := \frac{[-q^{4m};q^{64}]_\infty[q^{8n};q^{64}]_\infty(q^{64})_\infty^2}{[q^{4(m-n)};q^{64}]_\infty[-q^{4n};q^{64}]_\infty[q^{4(m+n)};q^{64}]_\infty}$$

and

$$A_N(m,n) = A_N(m,n;q) := \frac{[-q^{4n};q^{64}]_\infty[q^{8n};q^{64}]_\infty(q^{64})_\infty^2}{[q^{4(m-n)};q^{64}]_\infty[-q^{4m};q^{64}]_\infty[q^{4(m+n)};q^{64}]_\infty}$$

and set

$$U(m) = U(m;q) := (q^{64})_\infty^{-1}T_3(-q^{4m}, 1, q^{64}) \text{ and}$$

$$\mathcal{g}_*(m) = \mathcal{g}_*(m;q) := \mathcal{g}_*(-q^{4m}, q^{64}), \text{ for } * = M \text{ or } N.$$

Then, with the help of **1.4.8** and **1.2.0**, **1.5.0** becomes

$$S_N(3u+1, 8) = (-)^{u+1}(q)_\infty q^{-3u(u+1)/2+21}U(7-2u) - \mathcal{g}_N(7-2u)$$

$$+ (-)^u q^{-3u(u+1)/2}\sum_{\substack{r=0 \\ r\neq u}}^{3}(-)^r q^{3r(r+1)/2}A_N(7-2u, 7-2r) \qquad \mathbf{1.5.1}$$

In like manner, we find that

$$S_M(u,8) = -g_M(7-2u)$$

$$+ (-)^u q^{-u(u+1)/2} \sum_{\substack{r=0 \\ r\neq u}}^{3} (-)^r q^{r(r+1)/2} A_M(7-2u, 7-2r) \qquad 1.5.2$$

Suppose m is odd (or at least $\not\equiv 0 \bmod 8$) and set

$$X_N(m) = X_N(m;q) := H(q^{8m}; q^{64}) + H(-q^{4m}; q^{64})$$

$$\text{and } X_M(m) = X_M(m;q) := H(q^{8m}; q^{64}) - H(-q^{4m}; q^{64})$$

so that **1.2.13** and **1.2.3** read

$$3g_M(m) - g_M(3m) = X_M(m), \qquad 1.5.3$$

$$3g_N(m) - g_N(3m) = X_N(m) - 2. \qquad 1.5.4$$

Repeated applications of **1.5.3** now give

$$81g_N(m) - g_N(81m) = \sum_{i=0}^{3} 3^i X_N(3^{3-i}m) - 80, \qquad 1.5.5$$

while repeated applications of **1.2.1** give

$$g_N(m) - g_N(81m) = -15m \qquad 1.5.6$$

and, from **1.5.5** and **1.5.6**, we have

$$80g_N(m) = \sum_{i=0}^{3} 3^i X_N(3^{3-i}m) + 15m - 80. \qquad 1.5.7$$

In the same way, **1.5.2** and **1.2.11** give

$$80g_M(m) = \sum_{i=0}^{3} 3^i X_M(3^{3-i}m) + 5m. \qquad 1.5.8$$

I have so far defined $A_N(m,n)$ and $A_M(m,n)$ only for $m \not\equiv \pm n \bmod 16$. I now extend these definitions to the case $m = n$, defining

$$A_M(m,m) := m/16 - \mathcal{g}_M(m) = -\frac{1}{80}\sum_{l=0}^{3} 3^l X_M(3^{3-l}m) \quad \text{and}$$

$$A_N(m,m) := 3m/16 - 1 - \mathcal{g}_N(m) = -\frac{1}{80}\sum_{l=0}^{3} 3^l X_N(3^{3-l}m).$$

With these definitions, **1.5.7** and **1.5.8** show that **1.5.2** and **1.5.1** may be written as

$$S_M(u,8) = u/8 - 7/16 + (-)^u q^{-u(u+1)/2}\sum_{r=0}^{3}(-)^r q^{r(r+1)/2} A_M(7-2u, 7-2r) \qquad \mathbf{1.5.9}$$

and

$$S_N(3u+1,8) = 3u/8 - 5/16 + (-)^{u+1}(q)_\infty q^{-3u(u+1)/2+21}U(7-2u) + (-)^u q^{-3u(u+1)/2}\sum_{r=0}^{3}(-)^r q^{3r(r+1)/2} A_N(7-2u, 7-2r) \qquad \mathbf{1.5.10}$$

From **1.3.4**, **1.3.0** and **1.3.1** we have

$$N_8(0) = P\big(2S_N(0,8) + 1\big) = P\big(-2S_N(7,8) + 1\big),$$

$$N_8(1) = P\big(S_N(1,8) - S_N(0,8)\big) = P\big(S_N(1,8) + S_N(7,8)\big),$$

$$N_8(2) = P\big(S_N(2,8) - S_N(1,8)\big) = P\big(S_N(10,8) - S_N(1,8) - 1\big) - q^{-1},$$

$$N_8(3) = P\big(S_N(3,8) - S_N(2,8)\big) = -P\big(S_N(4,8) + S_N(10,8) - 1\big) + q^{-1},$$

$$N_8(4) = P\big(S_N(4,8) - S_N(3,8)\big) = 2PS_N(4,8),$$

and so **1.5.10** gives

$$N_8(0) = P/8 + 2q^{12}U(3) - 2P\sum_{r=0}^{3}(-)^r q^{3r(r+1)/2-9} A_N(3, 7-2r),$$

$$N_8(1) = P/8 - q^{21}U(7) - q^{12}U(3) + P\sum_{r=0}^{3}(-)^r q^{3r(r+1)/2}\big(A_N(7,7-2r) + q^{-9}A_N(3,7-2r)\big),$$

$$N_8(2) = P/8 + q^{21}U(7) + q^3U(1) - q^{-1}$$
$$- P\sum_{r=0}^{3}(-)^r q^{3r(r+1)/2}\left(q^{-18}A_N(1,7-2r) + A_N(7,7-2r)\right),$$

$$N_8(3) = P/8 - q^{18}U(5) - q^3U(1) + q^{-1}$$
$$+ P\sum_{r=0}^{3}(-)^r q^{3r(r+1)/2}\left(q^{-3}A_N(5.7-2r) + q^{-18}A_N(1.7-2r)\right).$$

$$N_8(4) = P/8 + 2q^{18}U(5) - 2P\sum_{r=0}^{3}(-)^r q^{3r(r+1)/2-3}A_N(5,7-2r). \qquad \mathbf{1.5.11}$$

Likewise, **1.3.4** and **1.5.9** give

$$M_8(0) = P/8 + P\sum_{r=0}^{3}(-)^r q^{r(r+1)/2}A_M(7,7-2r)$$

$$M_8(1) = P/8 - P\sum_{r=0}^{3}(-)^r q^{r(r+1)/2}\left(q^{-1}A_M(5.7-2r) + A_M(7.7-2r)\right)$$

$$M_8(2) = P/8 + P\sum_{r=0}^{3}(-)^r q^{r(r+1)/2}\left(q^{-3}A_M(3,7-2r) + q^{-1}A_M(5,7-2r)\right)$$

$$M_8(3) = P/8 - P\sum_{r=0}^{3}(-)^r q^{r(r+1)/2}\left(q^{-6}A_M(1.7-2r) + q^{-3}A_M(3.7-2r)\right)$$

$$M_8(4) = P/8 + 2P\sum_{r=0}^{3}(-)^r q^{r(r+1)/2-6}A_M(1,7-2r). \qquad \mathbf{1.5.12}$$

Note that, from **1.5.11** and **1.5.12**,

$$N_8(0) + 2N_8(1) + 2N_8(2) + 2N_8(3) + N_8(4) = P$$

and

$$M_8(0) + 2M_8(1) + 2M_8(2) + 2M_8(3) + M_8(4) = P$$

which is as it should be.

1.6 In this section, I find expressions along the lines of **1.5.11** for the $N_9(r)$.

We have

$$S_N(u,9) = \sum_n{}' (-)^n \frac{q^{n(3n+1)/2+un}}{1-q^{9n}}$$

$$= \sum_{r=-1}^{1} (-)^r q^{r(3r+1)/2+ur} \sum_m (-)^m \frac{q^{(9r+3u-12)m} q^{27m(m+1)/2}}{1-q^{9r}q^{27m}}$$

(with $\sum_m{}'$ for $\sum_m$ when r = 0)

$$= T_1^*(q^{3u-12}, q^{27}) - q^{1-u}T_1(q^{-9}, q^{3u-21}, q^{27})$$

$$- q^{2+u}T_1(q^9, q^{3u-3}, q^{27}) \qquad \textbf{1.6.0}$$

Set

$$B(m,n) = B(m,n;q) := \frac{[-q^{-9n};q^{27}]_\infty[-q^{3m-9n};q^{27}]_\infty(q^{27})_\infty}{[q^{-9n};q^{27}]_\infty[-q^{3m};q^{27}]_\infty[-1;q^{27}]_\infty},$$

for n not a multiple of 3, and

$$V(m) = V(m;q) := [-1;q^{27}]_\infty^{-1}(q^{27})^{-1}_\infty T_1(-q^{3m},-1,q^{27}),$$

$$Y(m) = Y(m;q) := H(-q^{3m};q^{27}),$$

$$h(m) = h(m;q) = h(q^{3m};q^{27})$$

With these definitions and the help of **1.2.4**, **1.6.0** becomes

$$S_N(u,9) = \Big([q^{12-3u};q^{27}]_\infty - q^{1-u}[q^{21-3u};q^{27}]_\infty$$

$$- q^{2+u}[q^{3-3u};q^{27}]_\infty\Big)(q^{27})_\infty V(4-u)$$

$$- h(4-u) + q^{1-u}B(4-u,-1) + q^{2+u}B(4-u,1). \qquad \textbf{1.6.1}$$

Now **1.2.16** shows that $h(0) = 0$ and then **1.2.15** shows that $h(27k) = k$ (for integer k). **1.2.18** and **1.2.16** now show that

$$9h(4 - u) = 3Y(4 - u) + Y(12 - 3u) + 4 - u. \qquad \mathbf{1.6.2}$$

Now, when n is a multiple of 3, set

$$B(m, n) := -\frac{1}{3} Y(m) - \frac{1}{9} Y(3m).$$

Then, with the help of **1.4.9** and **1.6.2**, **1.6.1** becomes

$$S_N(u, 9) = \frac{u - 4}{9} + \alpha_u (q)_\infty V(4 - u) + \sum_{n=-1}^{1} q^{n(3n+1)/2 + un} B(4 - u, n)) \qquad \mathbf{1.6.3}$$

So we have

$$N_9(0) = P/9 + 2V(4) + 2P\sum_{n=-1}^{1} q^{n(3n+1)/2} B(4, n),$$

$$N_9(4) = P/9 + q^{-2}V(1) - P\sum_{n=-1}^{1} q^{n(3n+1)/2 + 3n} B(1, n).$$

and, for k = 1, 2 and 3,

$$N_9(k) = P/9 + \alpha_k V(4 - k) - \alpha_{k-1} V(5 - k)$$
$$+ P\sum_{n=-1}^{1} q^{n(3n+1)/2 + (k-1)n}\left(q^n B(4 - k, n) - B(5 - k, n)\right)$$

I do not give similar expressions for the $M_9(r)$ because, at the time of writing, I have not been able to work them out. In any case, I do not need them for the proof of theorem E. The difficulty lies in the appearance of functions $T_{1/3}(., ., .)$ in the expressions for $S_M(u, 9)$ analogous to **1.5.0** and **1.6.0** and I have not been able to find transformation rules along the lines of **1.2.0** and **1.2.4** for such functions. Maybe there are similar transformation rules or maybe there are ways round this problem.

1.7 In the final section of this chapter, I find expressions for the $N_{12}(r)$. I do not do the same for the $M_{12}(r)$ for the reasons given at the end of the previous section.

For n not zero mod 4, set

$$C(m,n) = C(m,n;q) := \frac{[-q^{12n};q^{48}]_\infty [q^{2m-12n};q^{48}]_\infty (q^{48})^2_\infty}{[q^{12n};q^{48}]_\infty [q^{2m};q^{48}]_\infty [-1;q^{48}]_\infty}$$

and set

$$W(m) = W(m;q) := [-1;q^{48}]^{-1}_\infty (q^{27})^{-1}_\infty T_1(-q^{2m}, -1, q^{48}),$$

$$Z(m) = Z(m;q) := \frac{[q^{4m};q^{48}]^3_\infty (q^{48})^2_\infty}{[q^{2m};q^{48}]^3_\infty [q^{6m};q^{48}]_\infty},$$

$$h'(m) := h(-q^{2m};q^{48}).$$

We have

$$\begin{aligned} S_N(u,12) &= T^*(-q^{4u-22}, q^{48}) - q^{u+2}T_1(q^{12}, -q^{4u-10}, q^{48}) \\ &\quad + q^{2u+7}T_1(q^{24}, -q^{4u+2}, q^{48}) - q^{3u+15}T_1(q^{36}, -q^{4u+14}, q^{48}) \\ &= \Big([-q^{22-4u};q^{48}]_\infty - q^{u+2}[-q^{10-4u};q^{48}]_\infty + q^{2u+7}[-q^{-2-4u};q^{48}]_\infty \\ &\quad - q^{3u+15}[-q^{-14-4u};q^{48}]_\infty\Big)(q^{48})_\infty W(11-2u) \\ &\quad - h'(11-2u) + \sum_{n=1}^{3}(-)^n q^{n(3n+1)/2+un} C(11-2u,n) \end{aligned} \qquad \textbf{1.7.0}$$

Suppose m is odd (or at least not a multiple of 4). Then **1.2.18** gives

$$3h'(m) - h'(3m) = Z(m),$$

$$3h'(3m) - h'(9m) = Z(3m),$$

$$3h'(9m) - h'(27m) = Z(9m) \qquad \textbf{1.7.1}$$

and **1.2.15** gives

$$h'(27m) = h'(3m) + m. \qquad \textbf{1.7.2}$$

1.7.1 and **1.7.2** show that

$$24h'(m) = 8Z(m) + 3Z(3m) + Z(9m) + m \qquad \textbf{1.7.3}$$

and **1.7.3** gives

$$h'(11-2u) = 11/24 - u/12 - C(11-2u, 0) \qquad \textbf{1.7.4}$$

where

$$C(m, 0) := -\frac{1}{24}\Big(8Z(m) + 3Z(3m) + Z(9m)\Big).$$

Now **1.7.0**, **1.7.4** and **1.4.10** show that

$$S_N(u, 12) = u/12 - 11/24 + \alpha_u (q)_\infty W(11-u) + \sum_{n=0}^{3} (-)^n q^{n(3n+1)/2 + un} C(11-2u, n) \qquad \textbf{1.7.5}$$

and **1.7.5** with **1.3.4** gives

$$N_{12}(0) = P/12 + 2W(11) + 2P\sum(-)^n q^{n(3n+1)/2} C(11, n)$$

and

$$N_{12}(k) = P/12 + \alpha_k W(11-k) - \alpha_{k-1} W(12-k) + P\sum_{n=0}^{3}(-)^n q^{n(3n+1)/2 + (k-1)n}\Big(q^n C(11-2k, n) - C(13-2k, n)\Big) \qquad \textbf{1.7.6}$$

for $1 \le k \le 6$.