2. Modular Forms.

2.0 In this chapter, I discuss what I need from the theory of modular forms. A general reference is **[Ran]**, though I do things in a slightly different way.

If R is a commutative ring with identity, $SL_2(R)$ is the group of all 2×2 matrices with entries from R and determinant one. If S is also a commutative ring with identity, a homomorphism $\phi : R \longrightarrow S$ (that preserves the identity) induces in the obvious way a homomorphism $SL_2(\phi) : SL_2(R) \longrightarrow SL_2(S)$. In the fashionable jargon, SL_2 is a functor between the category of commutative rings with identity and the category of groups. Moreover, it is plain that SL_2 preserves products:

$$SL_2(R \times S) \simeq SL_2(R) \times SL_2(S). \qquad \textbf{2.0.0}$$

2.1 From now on, A will denote a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and, throughout this and the remaining sections of this chapter, N will denote an integer greater than two.

Define

$$\Gamma(N) := \left\{A \in SL_2(\mathbb{Z}) : A \equiv \pm I \bmod N\right\}$$

(which is $\bar{\Gamma}(N)$ in **[Ran]**), which is a normal subgroup of $SL_2(\mathbb{Z})$, and

$$\Gamma_0(N) := \left\{A \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod N\right\},$$

$$\Gamma_1(N) := \left\{A \in \Gamma_0(N) : a \equiv d \equiv \pm 1 \bmod N\right\},$$

which are subgroups (not normal) of $SL_2(\mathbb{Z})$.

I shall need to know the value of

$$\mu(N) := [SL_2(\mathbb{Z}) : \Gamma_1(N)].$$

(Here, [:] denotes the index of a subgroup in a group.) Denote by $\hat{\Gamma}_1(N)$ the image of $\Gamma_1(N)$ in $SL_2(\mathbb{Z}_N)$. Then

$$\mu(N) = [SL_2(\mathbb{Z}_N) : \hat{\Gamma}_1(N)]. \qquad \textbf{2.1.0}$$

Now the order of $\hat{\Gamma}_1(N)$ is easy to find. It is

$$\#\hat{\Gamma}_1(N) = 2N \qquad \textbf{2.1.1}$$

(since $N > 2$) and it remains for us to calculate $\#SL_2(\mathbb{Z}_N)$. For $n \in \mathbb{N}$, let

$$\alpha(n) := \#SL_2(\mathbb{Z}_n).$$

Lemma

$\alpha(mn) = \alpha(m)\alpha(n)$, if m and n are coprime, **2.1.2**

$\alpha(p^{k+1}) = p^3\alpha(p^k)$, if $k > 0$, **2.1.3**

$\alpha(n) = n^3 \prod_{p \mid n} (1 - 1/p^2)$ (the product being over the primes dividing n) **2.1.4**

Proof **2.1.2** follows from **2.0.0** and the fact that $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ when m and n are coprime.

2.1.3: Let K be the kernel of the map $SL_2(\mathbb{Z}_{p^{k+1}}) \longrightarrow SL_2(\mathbb{Z}_{p^k})$ induced from the projection $\mathbb{Z}_{p^{k+1}} \longrightarrow \mathbb{Z}_{p^k}$. Then

$$K = \left\{ \begin{pmatrix} 1 + rp^k & sp^k \\ tp^k & 1 + up^k \end{pmatrix} : 0 \le r, s, t < p \text{ and } u = 0, \text{ if } r = 0, = p - r, \text{ otherwise.} \right\}$$

Clearly, $\#K = p^3$, which gives us **2.1.3.**

2.1.4: After **2.1.2** and **2.1.3**, it remains to show that

$$\alpha(p) = p^3 - p.$$

The matrices in $SL_2(\mathbb{Z}_p)$ can be divided into three sets:

$$V_1 := \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \beta\gamma \neq 0, -1\right\},\ V_2 := \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \beta\gamma = -1\right\}$$

$$\text{and } V_3 := \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \beta\gamma = 0\right\}.$$

It's easy to see that

$$\#V_1 = (p - 1)^2(p - 2) \text{ and } \#V_2 = (p - 1)(2p - 1) = \#V_3,$$

so

$$\alpha(p) = (p - 1)^2(p - 2) + 2(p - 1)(2p - 1) = p^3 - p. \qquad \text{qed}$$

It now follows from **2.1.0, 2.1.1** and **2.1.4** that

$$\mu(N) = \frac{1}{2} N^2 \prod_{p|N} (1 - 1/p^2). \qquad \textbf{2.1.5}$$

2.2 Set

$$\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{im} \tau > 0\}, \qquad \mathbb{Q}^* := \mathbb{Q} \cup \{\infty\}.$$

$\mathbb{Q}^*$ is the set of *cusps.*

If $A \in SL_2(\mathbb{Z})$ and $\xi \in \mathbb{H}$ or $\mathbb{Q}^*$, it is easy to see that

$$A\xi := \frac{a\xi + b}{c\xi + d} \in \mathbb{H} \text{ (respectively, } \mathbb{Q}^*) \quad (A\infty = a/c.),$$

and $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ and (transitively) on $\mathbb{Q}^*$ in this way. A subgroup Γ of $SL_2(\mathbb{Z})$ induces equivalence relations on $\mathbb{H}$ and $\mathbb{Q}^*$, points ξ and ξ' being Γ-***equivalent*** if $\xi' = A\xi$, for some $A \in \Gamma$. I denote the equivalence class of ξ by $[\xi]_\Gamma$. The ***cusps*** of Γ are the members of the set $\Gamma\backslash\mathbb{Q}^*$ (the Γ-orbits of $\mathbb{Q}^*$).

Note that, if x and y are coprime integers and $A \in SL_2(\mathbb{Z})$, then $ax + by$ and $cx + dy$ are also coprime. So we can identify $\mathbb{Q}^*$ with the set

$$\left\{\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{Z}^2 : x \text{ and } y \text{ are coprime and } y \geq 0\right\}$$

(∞ corresponds to $\begin{pmatrix}1\\0\end{pmatrix}$) and the action of $SL_2(\mathbb{Z})$ on $\mathbb{Q}^*$ becomes matrix multiplication.

My aim in this section is to identify a complete set of distinct cusps of $\Gamma_1(N)$ For integers a, b and n, I write $a \equiv_n b$ for $a \equiv b \bmod n$.

Lemma

If x, y and n are integers with gcd(x, y, n) = 1, there are coprime integers x′ and y′ with $x' \equiv_n x$ *and* $y' \equiv_n y$. **2.2.0**

Proof Let $p_1, \ldots, p_k$ be the primes that divide both y and one of $x + rn$ ($-\infty < r < \infty$). In particular, suppose p_i divides both y and $x + r_i n$. The Chinese remainder theorem tells us that there's a number r such that $r \equiv r_i + 1 \bmod p_i$ for each $1 \leq i \leq k$. Take $x' = x + rn$, $y' = y$. qed

Lemma

For $\binom{x}{y}, \binom{x'}{y'} \in \mathbb{Q}^*$ *and* $n \in \mathbb{Z}$,

$$\left[\binom{x}{y}\right]_{\Gamma(n)} = \left[\binom{x'}{y'}\right]_{\Gamma(n)} \iff \binom{x}{y} \equiv_n \pm\binom{x'}{y'} \qquad \mathbf{2.2.1}$$

Proof $\Longrightarrow$ is obvious. For $\Longleftarrow$, I'll only treat the case $\binom{x}{y} \equiv_n \binom{x'}{y'}$ the other case being very similar. Suppose first that $\binom{x}{y} = \binom{1}{0}$. Since x′ and y′ are coprime, there are integers r and s such that

$$sx' - ry' = 1.$$

Since $x' \equiv_n 1$ and $y' \equiv_n 0$, it follows that $s \equiv_n 1$ and ∴ that the matrix

$$B = \begin{pmatrix} x' & r - rx' \\ y' & s - ry' \end{pmatrix}$$

lies in Γ(n) and $B\binom{1}{0} = \binom{x'}{y'}$. Now, for any $\binom{x}{y}$, choose $T \in SL_2(\mathbb{Z})$ such that $T\binom{x}{y} = \binom{1}{0}$. If $T\binom{x'}{y'} = \binom{x''}{y''}$, x″ and y″ are coprime and $\binom{1}{0} \equiv_n \binom{x''}{y''}$. So there's a matrix $B \in \Gamma(n)$ with $B\binom{1}{0} = \binom{x''}{y''}$. Now $C = T^{-1}BT$ is in Γ(n), since Γ(n) is normal, and $C\binom{x}{y} = \binom{x'}{y'}$. ▭ qed

For $n \in \mathbb{Z}$, let $\overline{m}$ denote the congruence class of $m \in \mathbb{Z}$ modulo n. Define

$$\mathbb{Q}^{(n)} := \left\{\binom{\overline{x}}{\overline{y}} \in \mathbb{Z}_n^2 : \overline{x} \text{ and } \overline{y} \text{ are coprime (in } \mathbb{Z}_n)\right\}$$

(Note that $\overline{x}$ and $\overline{y}$ are coprime in $\mathbb{Z}_n$ exactly when gcd(x, y, n) = 1.) $SL_2(\mathbb{Z})$ acts on $\mathbb{Q}^{(n)}$, as on $\mathbb{Q}^*$, by matrix multiplication and there's a map $\mathbb{Q}^* \xrightarrow{\alpha} \mathbb{Q}^{(n)}$ (reduce components modulo n) which, by **2.2.0**, is surjective. α commutes with the action of $SL_2(\mathbb{Z})$ and so, for any subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, induces a map $\Gamma\backslash\alpha : \Gamma\backslash\mathbb{Q}^* \longrightarrow \Gamma\backslash\mathbb{Q}^{(n)}$. **2.2.1** shows that $\Gamma\backslash\alpha$ is injective, when $\Gamma(n) \subseteq \Gamma$, and so we have

Lemma

Suppose $\Gamma(n) \subseteq \Gamma \subseteq SL_2(\mathbb{Z})$. Then $\Gamma\backslash\alpha$ is an isomorphism between the set of cusps of Γ and $\Gamma\backslash\mathbb{Q}^{(n)}$. ▣ **2.2.2**

(This result appears in [S, p.11], though the author inadvertently fails to mention the "mod n" aspect.)

I now describe the set of cusps of $\Gamma_1(N)$.

Theorem

Suppose $N > 2$ and define

$$C^{(N)} = \bigcup_{d \mid N} C_d^{(N)}$$

where

$$C_d^{(N)} = \Big\{ \Big[\binom{x}{y}\Big]_{\Gamma_1(N)} \in \Gamma_1(N)\backslash\mathbb{Q}^{(N)} : gcd(y, N) = d$$
$$\text{and} \quad 0 < x < \tfrac{1}{2}d,\ 0 \le y < N$$
$$\text{or} \quad x = \tfrac{1}{2}d,\ 0 \le y \le \tfrac{1}{2}N$$
$$\text{or} \quad x = 0,\ 0 < y < \tfrac{1}{2}N \Big\}.$$

Then $C^{(N)}$ may be identified (via the map $\Gamma_1(N)\backslash\alpha$ of 2.2.2) with the set of cusps of $\Gamma_1(N)$. **2.2.3**

Proof Suppose $\binom{\bar{x}}{\bar{y}} \in \mathbb{Q}^{(N)}$ and $\gcd(y, N) = d$. The subgroup of $\mathbb{Z}_N$ generated by y has smallest generator $\gcd(N, y) = d$. This means that there are unique integers $x' = \varepsilon x + ry + sN$ and $y' = \varepsilon y + tN$ ($\varepsilon = \pm 1$. r, s and t integers) such that

$$0 < x' < \frac{1}{2}d \quad \text{and} \quad 0 \le y' < N$$

$$\text{or} \quad x' = 0,\ 0 < y' < \frac{1}{2}N$$

$$\text{or} \quad x' = \frac{1}{2}d \text{ and } 0 \le y' \le \frac{1}{2}N$$

Plainly gcd$(y', N) = d$. Now the matrix $\begin{pmatrix} \varepsilon & r \\ 0 & \varepsilon \end{pmatrix}$ lies in $\Gamma_1(N)$ and carries $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$ to $\begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix}$ and so each member of $\Gamma_1(N)\backslash\mathbb{Q}^{(N)}$ has a representative with the stated properties. The uniqueness of such a representative follows from the uniqueness of the integers x' and y' cited above. qed

2.3 Suppose G is a group acting on a set S. For $s \in S$, let G_s denote the subgroup of G of elements that fix s. Suppose H is a subgroup of G. If $s \in S$, define the ***width of s relative to H*** to be

$$w(s\,;H) := [G_s : H_s].$$

Note that, if s and t are H-equivalent, $w(s\,;H) = w(t\,;H)$. So we can talk about the width of an orbit of H.

Now suppose G acts transitively on S and let $T \subseteq S$ be a complete set of distinct representatives of $H\backslash S$. While I don't actually need the next result, I have found it useful, for example in checking that I have found all the cusps of a subgroup of $SL_2(\mathbb{Z})$. In this lemma, $[G : H]$ denotes ambiguously the set of right cosets of H in G and the number thereof.

Lemma

With the above hypothesis and notation, there is a bijection

$$\alpha : [G : H] \longrightarrow \bigcup_{t \in T} [G_t : H_t].$$

As a consequence, if $[G:H]$ is finite, then each $w(s;H)$ is finite and

$$\sum_{\zeta \in H \backslash S} w(\zeta;H) = [G:H].$$

Proof Pick $* \in S$ and, for each $s \in S$, pick $x_s \in G$ with $x_s* = s$ and so that $x_s x_{s'}^{-1}$ lies in H whenever s and s′ are in the same orbit of H. Such a choice is plainly always possible. I define α by

$$\alpha(Hg) := H_t x_t x_s^{-1} g x_t^{-1},$$

where $g* = s = ht$ with $t \in T$ and $h \in H$. Then α is a well-defined bijection, with inverse

$$H_t g_t \longmapsto H g_t x_t$$

(where $g_t \in G_t$). qed

α is, of course, far from natural.

$SL_2(\mathbb{Z})$ acts transitively on $\mathbb{Q}^*$. If Γ is a subgroup of $SL_2(\mathbb{Z})$ and $\zeta \in \Gamma\backslash\mathbb{Q}^*$ is a cusp of Γ, the width of ζ relative to Γ is $w(\zeta;\Gamma)$, as defined above. Now $SL_2(\mathbb{Z})_\infty$ is the group $\langle U, -U\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}$, where $U := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, so, if $\zeta \in \mathbb{Q}^*$ and $\zeta = T\infty$, with $T \in SL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})_\zeta = \langle TUT^{-1}, -TUT^{-1}\rangle$. If Γ contains $-I$, it is plain that

$$w(\zeta;\Gamma) = \text{the smallest positive integer } k \text{ with } TU^kT^{-1} \in \Gamma. \qquad \textbf{2.3.0}$$

I now give the widths of the various cusps of $\Gamma_1(N)$. Recall **2.2.3**, that the cusps of $\Gamma_1(N)$ lie in various sets $C_d^{(N)}$, where $d|N$.

Lemma

If $N \neq 4$, each cusp in $C_d^{(N)}$ has width N/d. The same is true when $N = 4$, except that the cusp $1/2$ has width 1. **2.3.1**

Proof Take $N \neq 4$. Suppose $\zeta = \binom{x}{y} \in \mathbb{Q}^*$ is (a representative of) a cusp of $\Gamma_1(N)$ lying in (identified with) a member of $C_d^{(N)}$. Then $\gcd(y, N) = d$. Choose a matrix $T = \begin{pmatrix} x & u \\ y & v \end{pmatrix} \in SL_2(\mathbb{Z})$ carrying ∞ to ζ. Suppose

$$TU^kT^{-1} = \begin{pmatrix} \varepsilon-kxy & kx^2 \\ -ky^2 & \varepsilon+kxy \end{pmatrix}$$

($\varepsilon = \pm 1$) lies in $\Gamma_1(N)$. Then $N | ky^2$ and $N | kxy$, which means (since x and y are coprime) that $N | ky$. It follows that $N/d | k$. On the other hand, it is clear that $TU^{n/d}T^{-1} \in \Gamma_1(N)$. The exceptional case when $N = 4$ is easily verified. qed

2.4 The ***Dedekind eta function***, η, is defined on $\mathbb{H}$ by

$$\eta(\tau) := \exp(\pi i\tau/12)\prod_{n=1}^{\infty}\left(1 - \exp(2\pi i n\tau)\right)$$
$$= q^{1/24}(q)_{\infty},$$

where $q := \exp(2\pi i\tau)$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have **[Kno, thm 2.2, p. 51]**

$$\eta(A\tau) = e(A;\tau)\eta(\tau) = \varepsilon(A)\sqrt{c\tau+d}\ \eta(\tau), \qquad \textbf{2.4.0}$$

for $\tau \in \mathbb{H}$, where $\varepsilon(A)$ is a certain 24th. root of unity (independent of τ). To give the value of $\varepsilon(A)$, define the symbols

$$\left(\frac{c}{d}\right)^* := \left(\frac{c}{|d|}\right)$$

$$\left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right), \text{ if } c \geq 0 \text{ or } d > 0, \ -\left(\frac{c}{|d|}\right) \text{ otherwise}$$

for coprime integers c and d, with d odd, where $\left(\frac{\bullet}{\bullet}\right)$ is the ***Legendre-Jacobi*** symbol [**H+W. §6.5**]. Then

$$\varepsilon(A) = \begin{cases} \left(\frac{d}{c}\right)^* \exp(\pi i/12)\big((a+d)c + bd(1-c^2) - 3c\big), & c \text{ odd}, \\ \left(\frac{c}{d}\right)_* \exp(\pi i/12)\big((a+d)c + bd(1-c^2) + 3d(1-c) - 3\big), & d \text{ odd}. \end{cases} \qquad \mathbf{2.4.1}$$

(taking the value of the square root with $-\pi/2 \leq \arg\sqrt{c\tau + d} < \pi/2$).

Suppose that f is any complex valued function defined on $\mathbb{H}$. For k an integer and $A \in SL_2(\mathbb{Z})$, define the function $(f|_k A)$ on $\mathbb{H}$ by

$$(f|_k A)(\tau) := e(A;\tau)^{-k} f(A\tau).$$

It follows easily from **2.4.0** that, if B is also in $SL_2(\mathbb{Z})$,

$$(f|_k AB) = \big((f|_k A)|_k B\big) \qquad \mathbf{2.4.2}$$

2.4.0 also shows that, with $U := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$e(U^h;\tau) = u, \text{ a 24th root of unity.} \qquad \mathbf{2.4.3}$$

(in fact, from **2.4.1**, $u = \exp(\pi i h/12)$).

Now suppose Γ is a subgroup of $SL_2(\mathbb{Z})$, of finite index and that f is a function meromorphic on $\mathbb{H}$ that also satisfies

$$(f|_k A) = \chi_f(A) f \qquad \mathbf{2.4.4}$$

for every $A \in \Gamma$, where each $\chi_f(A)$ is a root of unity. (If **2.4.4** holds, then, as the notation suggests, χ_f is a character on Γ. I leave the proof as an easy exercise for the reader.) Suppose ζ is a cusp of Γ of width h and take $T \in SL_2(\mathbb{Z})$ with $T\infty = \zeta$. Then

$$\begin{aligned}
(f|_k T)(\tau + h) &= (f|_k T)(U^h\tau) \\
&= u^k\big((f|_k T)|_k U^h\big) && \text{(with u as at } \mathbf{2.4.3}) \\
&= u^k(f|_k TU^h)(\tau) && \text{(by } \mathbf{2.4.2}) \\
&= u^k(f|_k VT)(\tau) && \text{(where } V = TU^hT^{-1} \in \Gamma, \text{ by } \mathbf{2.3.0}) \\
&= u^k\big((f|_k V)|_k T\big)(\tau) && (\mathbf{2.4.2} \text{ again}) \\
&= u^k\chi_f(A)(f|_k T)(\tau)
\end{aligned} \qquad \mathbf{2.4.5}$$

If we now set $u^k\chi_f(V) = \exp(2\pi i r)$ $(0 \le r < 1)$ and define the function g by

$$g(\tau) := \exp(-2\pi i r\tau/h)(f|_k T)(\tau),$$

2.4.5 shows that g is invariant under $\tau \mapsto \tau + h$. Now suppose g is holomorphic in a region $\operatorname{im}\tau > \delta > 0$ (equivalently, f is holomorphic inside a horocycle at ζ). Then g has a Fourier series expansion

$$g(\tau) = \sum_n a_n q_h^n ,$$

where $q_h := \exp(2\pi i\tau/h)$, and $\therefore$

$$(f|_k T)(\tau) = q_h^r \sum_n a_n q_h^n . \qquad \mathbf{2.4.6}$$

f is *meromorphic* at the cusp ζ if f is holomorphic inside a horocycle at ζ and if the series in **2.4.6** is finite on the left. If this is so and if m is the smallest integer with $a_m \neq 0$, the *order* of f at ζ (with respect to Γ) is

$$\mathrm{ord}(\mathfrak{f}, \zeta, \Gamma) := r + m.$$

This number is independent of the choice of T with $T\infty = \infty$ (a different choice merely multiplies **2.4.6** by a root of 1) and furthermore

$$\mathrm{ord}(\mathfrak{f}, \xi, \Gamma) = \mathrm{ord}(\mathfrak{f}, \zeta, \Gamma),$$

when ζ and ξ are Γ-equivalent. So we can speak of the order of $\mathfrak{f}$ at a cusp of Γ.

I now define a ***modular form of weight k/2*** on Γ to be a function that satisfies **2.4.4** and is meromorphic on $\mathbb{H}$ and at the cusps. So η is a modular form of weight 1/2 on $SL_2(\mathbb{Z})$ ($SL_2(\mathbb{Z})$ has just one cusp at ∞ and η is visibly meromorphic there). The following result, a simple corollary of a theorem due to Rankin **[Ran, thm. 4.1.4]** is the linch-pin of my proofs of theorems D, E and H:

Theorem

> *Suppose Γ is a subgroup of $SL_2(\mathbb{Z})$ of index μ and that Γ contains -I. If $\mathfrak{f}$ is a modular form of weight r on Γ that is holomorphic on $\mathbb{H}$ and not identically zero then,*

$$\sum_{\alpha \in \Gamma\backslash\mathbb{Q}^*} \mathrm{ord}(\mathfrak{f}, \alpha, \Gamma) \le \mu r/12. \qquad \text{qed} \qquad \mathbf{2.4.7}$$

2.5 For positive integers n, define functions $\eta(n) = \eta(n\,;\tau)$ on $\mathbb{H}$ by:

$$\eta(n\,;\tau) := \eta(n\tau) = \eta(L_n\tau),$$

where $L_n := \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$. For $A \in \Gamma_0(n)$, define

$$A^{(n)} := L_n A L_n^{-1} = \begin{pmatrix} a & nb \\ c/n & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Now, for $A \in \Gamma_o(n)$,

$$\eta(n\,;A\tau) = \eta(L_n A\tau) = \eta(A^{(n)}L_n\tau) = \varepsilon(A^{(n)})\sqrt{c\tau+d}\;\eta(n\,;\tau),$$

so

$$\left(\eta(n)\mid_1 A\right) = \upsilon\left(\eta(n)\,;A\right)\eta(n)$$

where

$$\upsilon\left(\eta(n)\,;A\right) = \varepsilon(A^{(n)})\varepsilon(A)^{-1} \qquad 2.5.0$$

and $\eta(n)$ is a modular form of weight 1/2 on $\Gamma_o(n)$

Rademacher **[Rad, 81.2, p.181]** defines functions

$$\Theta_{\mu,\nu}(v\mid\tau) := \sum_{n=-\infty}^{\infty}(-)^{\nu n}\exp\left((n+\mu/2)^2\pi i\tau\right)\exp\left(2\pi i(n+\mu/2)v\right)$$

and shows that

$$\Theta_{1,1}(v\mid A\tau) = \varepsilon(A)^3\exp\left(\pi icv^2(c\tau+d)\right)\sqrt{c\tau+d}\;\Theta_{1,1}\left(v(c\tau+d)\mid\tau\right) \qquad 2.5.1$$

$$\Theta_{1,0}(v\mid A\tau) = i^{1-d}\exp(\pi icd/4)\varepsilon(A)^3\exp\left(\pi icv^2(c\tau+d)\right)$$
$$\times\sqrt{c\tau+d}\;\Theta_{1-c,1-d}\left(v(c\tau+d)\mid\tau\right) \qquad 2.5.2$$

(Here, Rademacher has $\sqrt{(c\tau+d)/i}$ where I have $\sqrt{c\tau+d}$, so his expressions look a little different.)

Note the obvious

$$\Theta_{\mu+2,\nu}(v\mid\tau) = (-)^{\nu}\Theta_{\mu,\nu}(v\mid\tau), \qquad \Theta_{\mu,\nu+2}(v\mid\tau) = \Theta_{\mu,\nu}(v\mid\tau)$$

(so that really there are only four functions $\Theta_{\mu,\nu}$) and

$$\Theta_{1,1}(v+1\mid\tau) = -\Theta_{1,1}(v\mid\tau), \qquad \Theta_{1,0}(v+1\mid\tau) = -\Theta_{1,0}(v\mid\tau). \qquad 2.5.3$$

For integers $n > 0$ and k, define functions $s_n(k)$ and $t_n(k)$ on $\mathbb{H}$ by

$$s_n(k;\tau) := -\exp(\pi i k^2\tau/n)\Theta_{1,1}(k\tau \mid n\tau) = q^{(n/2-k)^2/2n}[q^k;q^n]_\infty(q^n)_\infty$$

$$t_n(k;\tau) := \exp(\pi i k^2\tau/n)\Theta_{1,0}(k\tau \mid n\tau) = q^{(n/2-k)^2/2n}[-q^k;q^n]_\infty(q^n)_\infty$$

These functions $s_n(k)$ and $t_n(k)$ and the fact that, as I shall soon show, they are modular forms on $\Gamma_1(n)$ and $\Gamma_1(2n)$ respectively must be well known, but I've found no reference to them in the published literature. I got the idea from **[Gar2′]**, in which the author introduces certain special cases of these functions (see §**0.5**).

Note that

$$s_n(-k) = -s_n(k) = s_n(k+n) \qquad \textbf{2.5.4}$$

$$t_n(-k) = t_n(k) = t_n(k+n) \qquad \textbf{2.5.5}$$

and that

$$\exp(\pi i k^2\tau/n)\Theta_{0,0}(k\tau \mid n\tau) = t_n(n/2-k;\tau) \qquad \textbf{2.5.6}$$

For a function f of x of the form $f(x) = x^r(a_0 + a_1x + a_2x \ + \ldots)$ with $a_0 \neq 0$ and r rational, define: $\operatorname{ord} f := r$. Now, writing $\lambda_n(k)$ for the least nonnegative residue of k modulo n, **2.5.4** and **2.5.5** and the definitions of these functions show that

$$\operatorname{ord} s_n(k) = \left(n/2 - \lambda_n(k)\right)^2/2n = \operatorname{ord} t_n(k) \qquad \textbf{2.5.7}$$

Now, if $A \in \Gamma_0(n)$, we have, with the help of **2.5.1**,

$$s_n(k;A\tau) = -\exp(\pi i k^2 A\tau/n)\Theta_{1,1}(kA\tau \mid L_n A\tau)$$

$$= -\exp(\pi i k^2 A\tau/n)\Theta_{1,1}(kA\tau \mid A^{(n)}L_n\tau)$$

$$= -\varepsilon(A_n)^3\exp(\pi i k^2 A\tau/n)\exp\left(\pi i(kA\tau)^2(c/n)(c\tau+d)\right)\sqrt{c\tau+d}$$

$$\times\ \Theta_{1,1}\left(kA\tau(c\tau+d) \mid L_n\tau\right)$$

$$= -\varepsilon(A_n)^3\exp\left(\pi i(k^2A\tau/n)(1 + c(a\tau+b)\right)\sqrt{c\tau+d}\ \Theta_{1,1}\left(k(a\tau+b) \mid n\tau\right)$$

$$= -\varepsilon(A_n)^3\exp\left(\pi i k^2 a(a\tau+b)/n\right)\sqrt{c\tau+d}\ (-)^{kb}\Theta_{1,1}(ka\tau \mid n\tau)$$

(where I've used **2.5.3** and the fact that $1+bc = ad$)

$$= (-)^{kb}\varepsilon(A^{(n)})^3\exp(\pi i k^2 ab/n)\sqrt{c\tau+d}\ s_n(ak;\tau). \qquad \textbf{2.5.8}$$

Thus, for $A \in \Gamma_0(n)$,

$$\left(s_n(k) \mid_1 A\right) = \upsilon\left(s_n(k);A\right)s_n(ak), \qquad \textbf{2.5.9}$$

where

$$\upsilon\left(s_n(k);A\right) = (-)^{kb}\exp(\pi i k^2 ab/n)\varepsilon(A^{(n)})^3\varepsilon(A)^{-1}. \qquad \textbf{2.5.10}$$

Working in the same way with **2.5.2**, we have, for $A \in \Gamma_0(n)$, that

$$\left(t_n(k) \mid_1 A\right) = \upsilon(t_n(k);A)t_n(ak),\ \text{if}\ A \in \Gamma_0(2n)$$

$$= \upsilon(t_n(k);A)t_n(n/2 - ak),\ \text{if}\ A \notin \Gamma_0(2n)\ \text{and}\ n\ \text{is even.} \qquad \textbf{2.5.11}$$

where

$$\upsilon(t_n(k);A) = (-)^{kb}i^{1-d}\exp(\pi icd/4n)\exp(\pi i k^2 ab/n)\varepsilon(A^{(n)})^3\varepsilon(A)^{-1} \qquad \textbf{2.5.12}$$

2.5.4, **2.5.5**, **2.5.9** and **2.5.11** show that $s_n(k)$ and $t_n(k)$ are modular forms of weight 1/2 on $\Gamma_1(n)$, respectively $\Gamma_1(2n)$ (that these functions are meromorphic at the cusps will be shown in the next section).

For an α-tuple $\mathbf{i} = (i_0, \ldots, i_{\alpha-1})$ with integral entries, set $\operatorname{len}\mathbf{i} = \alpha$ and define

$$s_n(\mathbf{i}) = \prod_{r=0}^{\alpha-1} s_n(i_r).$$

Define $t_n(\mathbf{i})$ and $\eta(\mathbf{i})$ in the same way and, for even n, define

$$t'_n(\mathbf{i}) = \prod_{r=0}^{\alpha-1} t_n(n/2 - i_r).$$

In sections **3.1-3** I shall be dealing with functions of the form

$$\mathcal{X}(\mathbf{m}) = \frac{s_n(\mathbf{u})t_n(\mathbf{v})\eta(\mathbf{b})}{s_n(\mathbf{x})t_n(\mathbf{y})\eta(\mathbf{c})} \qquad \textbf{2.5.13}$$

in which **u**, **v**, **x** and **y** are linearly dependent on the parameters **m** and in which entries of **b** and of **c** (which will be independent of **m**) are positive integers dividing n. Set

$$w := \operatorname{len}\mathbf{u} + \operatorname{len}\mathbf{v} + \operatorname{len}\mathbf{b} - \operatorname{len}\mathbf{x} - \operatorname{len}\mathbf{y} - \operatorname{len}\mathbf{c}. \qquad \textbf{2.5.14}$$

Then, for $A \in \Gamma_0(n)$ (n even), we have, by **2.5.9** and **2.5.11**

$$\begin{aligned}\left(\mathcal{X}(\mathbf{m}) \mid_w A\right) &= \upsilon\left(\mathcal{X}(\mathbf{m}); A\right)\mathcal{X}(a\mathbf{m}), \text{ if } A \in \Gamma_0(2n),\\ &= \upsilon\left(\mathcal{X}(\mathbf{m}); A\right)\mathcal{X}'(a\mathbf{m}), \text{ if } A \notin \Gamma_0(2n),\end{aligned} \qquad \textbf{2.5.15}$$

where

$$\upsilon\left(\mathcal{X}(\mathbf{m}); A\right) = \upsilon\left(s_n(\mathbf{u}); A\right)\upsilon\left(t_n(\mathbf{v}); A\right)\upsilon\left(\eta(\mathbf{b}); A\right) \times \upsilon\left(s_n(\mathbf{x}); A\right)^{-1}\upsilon\left(t_n(\mathbf{y}); A\right)^{-1}\upsilon\left(\eta(\mathbf{b}); A\right)^{-1}$$

and $\mathcal{X}'(a\mathbf{m})$ is $\mathcal{X}(a\mathbf{m})$, with t'_n for t_n.

I shall call a function like $\mathcal{X}(\mathbf{m})$ at **2.5.13** or $\mathcal{X}'(\mathbf{m})$ a ***theta product*** of ***index*** n, whose ***weight*** is w/2 (w as defined at **2.5.14**). If $\mathcal{X}(\mathbf{m})$ and $\mathcal{Y}(\mathbf{m})$ are theta products of the same weight and same index (n, say), and if $\upsilon\left(\mathcal{X}(\mathbf{m}); A\right) =$

$\upsilon(\mathcal{Y}(m)\,;A)$ for all $A \in \Gamma_0(2n)$, I shall say that $\mathcal{X}(m)$ and $\mathcal{Y}(m)$ are ***compatible***.

Suppose $\mathcal{X}(m)$ and $\mathcal{Y}(m)$ are compatible. Then $\mathcal{Q}(m) = \mathcal{X}(m) + \mathcal{Y}(m)$ transforms according to

$$\left(\mathcal{Q}(m)\,|_w\, A\right) = \upsilon\left(\mathcal{Q}(m)\,;A\right)\mathcal{Q}(am),$$

for every $A \in \Gamma_0(2n)$, where $\upsilon\left(\mathcal{Q}(m)\,;A\right) = \upsilon\left(\mathcal{X}(m)\,;A\right)$.

Define the ***sign*** of the theta product $\mathcal{X}(m)$ to be $(-)^{\mathrm{len}\,\mathbf{u} + \mathrm{len}\,\mathbf{x}}$ and, for numbers $a \equiv \pm 1 \bmod n$, define $\chi_n(a) = \pm 1$ by:

$$\chi_n(a) \equiv a \bmod n. \qquad \textbf{2.5.16}$$

Then I say that that theta products $\mathcal{X}(m)$ and $\mathcal{Y}(m)$ are ***coherent*** if they are compatible, when they have the same sign, and otherwise

$$\upsilon\left(\mathcal{X}(m)\,;A\right) = \chi_{2n}(a)\upsilon\left(\mathcal{Y}(m)\,;A\right).$$

for every $A \in \Gamma_1(2n)$. It is plain that a sum of mutually coherent theta products of index n and weight w/2 is a modular form of weight w/2 on $\Gamma_1(2n)$.

2.6 In this section, I calculate the orders of the forms $\eta(r)$, for $r\,|\,n$, $s_n(k)$ and $t_n(k)$ at the various cusps of $\Gamma_1(2n)$.

Suppose that $\zeta = x/y \in \mathbb{Q}^*$ with $\gcd(y, 2n) = \delta$, so ζ represents a cusp in $C_{\delta}^{(2n)}$. Take a matrix $T = \begin{pmatrix} x & * \\ y & z \end{pmatrix} \in SL_2(\mathbb{Z})$ that carries ∞ to ζ. Suppose $r|n$ and set $\delta' = \gcd(r, y)$, $r' = r/\delta'$, $y' = y/\delta'$. Let $T^* := \begin{pmatrix} r'x & * \\ y' & z' \end{pmatrix} \in SL_2(\mathbb{Z})$. Then

$$T^{*-1}L_rT = V := \begin{pmatrix} \delta' & * \\ 0 & r' \end{pmatrix},$$

where $\delta' = \gcd(y, r)$, and ∴

$$\eta(r;T\tau) = \eta(T^*V\tau) = \varepsilon(T^*)\sqrt{(y\tau+z)/r'}\,\eta(V\tau)$$

and (since, by **2.3.1**, cusps in $\mathcal{C}_\delta^{(2n)}$ have width $2n/\delta$) it follows that

$$\operatorname{ord}\big(\eta(r),\ \zeta,\ \Gamma_1(n)\big) = n\delta'^2/12r\delta. \qquad \textbf{2.6.0}$$

Now calculations similar to those preceding **2.5.8** show that

$$s_n(k;T\tau) = u\sqrt{(y\tau+z)/n'}\,\exp\big(\pi ik^2x(x\tau+*)/n\big)\Theta_{1,1}\big(k(x\tau+*)/n' \mid (\delta'^2\tau+*)/n\big)$$

where $u \in \mathcal{U}$, the $*$'s are (unimportant) integers and δ', n' (and y' below) are as above,with n for r. It follows from **2.5.7** and **2.3.1** that

$$\begin{aligned}\operatorname{ord}\big(s_n(k),\ \zeta,\ \Gamma_1(2n)\big) &= \operatorname{ord} s_{\delta'^2/n}(kx/n') \times (2n/\delta)\\ &= (\delta'/2 - \lambda)^2/\delta,\end{aligned} \qquad \textbf{2.6.1}$$

where $\lambda = \lambda_{\delta'}(kx)$ is the least nonnegative residue of kx mod δ'.

In the same way, we have

$$t_n(k;T\tau) = u\sqrt{(y\tau+z)/n'}\,\exp\big(\pi ik^2x(x\tau+z)/r\big)\Theta_{1-y',1-z'}\big(k(x\tau+z)/n' \mid (\delta'^2\tau+*)/n\big),$$

We can suppose z' is odd and then

$$\begin{aligned}\operatorname{ord}(t_n(k),\ \zeta,\ \Gamma_1(2n) &= \operatorname{ord} t_{\delta'^2/n}(kx/n')\\ &= (\delta'/2 - \lambda)^2/\delta \qquad \text{if } y' \text{ is even,}\\ &= \operatorname{ord} t_{\delta'^2/n}(\delta'^2/2n - kx/n')\\ &= \mu^2/\delta, \qquad \text{if } y' \text{ is odd,}\end{aligned} \qquad \textbf{2.6.2}$$

where λ is as above and μ satisfies $kx \equiv \mu$ mod δ' and $-\delta'/2 < \mu \le \delta'/2$.