3. The proofs

3.0 In this section, I prove the "$\overset{6}{=}$"'s of theorem E. I restate them here:

Theorem

$$N(3,9,3n) = N(4,9,3n). \qquad \textbf{3.0.0}$$

$$N(1,9,3n+1) + N(2,9,3n+1) = N(3,9,3n+1) + N(4,9,3n+1). \qquad \textbf{3.0.1}$$

$$N(0,9,3n+2) = N(4,9,3n+2). \qquad \textbf{3.0.2}$$

Proof I prove **3.0.0**, **3.0.1** and **3.0.2** by showing that

$$N_9(4) - N_9(3) = -q^{-1}V(6) + 2q^{-2}V(3) - q^{-2}\frac{[-q^9;q^{27}]_\infty^2(q^{27})_\infty}{[q^{12};q^{27}]_\infty[q^3;q^{27}]_\infty[-q^3;q^{27}]_\infty^2} + q^{-1}\frac{[-q^{12};q^{27}]_\infty(q^{27})_\infty}{[q^{12};q^{27}]_\infty[q^6;q^{27}]_\infty[-1;q^{27}]_\infty} \qquad \textbf{3.0.3}$$

$$N_9(4) + N_9(3) - N_9(2) - N_9(1) = V(12) + 2q^{-1}V(6) - q^{-1}\frac{[-q^9;q^{27}]_\infty^2(q^{27})_\infty}{[q^3;q^{27}]_\infty[q^6;q^{27}]_\infty[-q^6;q^{27}]_\infty^2} - \frac{[-q^3;q^{27}]_\infty(q^{27})_\infty}{[q^3;q^{27}]_\infty[q^{12};q^{27}]_\infty[-1;q^{27}]_\infty} \qquad \textbf{3.0.4}$$

$$N_9(0) - N_9(4) = -q^{-2}V(3) + 2V(12) - q^3\frac{[-q^9;q^{27}]_\infty^2(q^{27})_\infty}{[q^3;q^{27}]_\infty[q^6;q^{27}]_\infty[-q^6;q^{27}]_\infty^2} - \frac{[-q^6;q^{27}]_\infty(q^{27})_\infty}{[q^6;q^{27}]_\infty[q^3;q^{27}]_\infty[-1;q^{27}]_\infty} \qquad \textbf{3.0.5}$$

I shall only prove **3.0.3**. The proofs of **3.0.4** and **3.0.5** are much the same. From **1.3.4** we have (suppressing subscripts)

$$N_9(0) - N_9(4) = P\left(S_N(3,9) + 2S_N(0,9) + 1\right). \qquad \textbf{3.0.6}$$

Now **1.6.1** shows that

$$S_N(3,9) + 2S_N(0,9) + 1 - P^{-1}\left(-q^{-2}V(3) + 2V(12)\right)$$
$$= -h(1) - 2h(4) + 1 + q^{-2}B(1,-1) + q^{5}B(1,1)$$
$$+ 2qB(4,-1) + 2q^{2}B(4,1), \qquad \textbf{3.0.7}$$

and **1.2.16**, **1.2.15** and **1.2.14** give

$$-h(1) - 2h(4) + 1 = -\frac{[q^{12};q^{27}]_\infty^2\,[q^3;q^{27}]_\infty(q^{27})_\infty^2}{[-q^{12};q^{27}]_\infty^2\,[-q^3;q^{27}]_\infty[-1;q^{27}]_\infty} \qquad \textbf{3.0.8}$$

Now, multiplying the (putative) identity **3.0.4** by $P^{-1} = (q)_\infty$ (in the form **1.4.3**) and using **3.0.6**, **3.0.7** and **3.0.8**, we find that, to establish **3.0.4**, we must establish the three identities

$$-\frac{[q][q^4]^2}{[-q][-q^4]^2[-1]} = q\frac{[-q^3]^2}{[q^2][-q^4]^2} - \frac{[-q^2]}{[q^2][-1]} \qquad \textbf{3.0.9}$$

$$q^{-1}\frac{[-q^3][-q^4]}{[q^3][-q][-1]} + 2\frac{[-q^3][-q^2]}{[q^3][-q^4][-1]}$$
$$= q^{-1}\frac{[-q^2][q^4]}{[q][q^2][-1]} - q\frac{[-q^3]^2}{[q^4][-q^4]^2} \qquad \textbf{3.0.10}$$

$$q^{-1}\frac{[-q^3][-q^2]}{[q^3][-q][-1]} + 2\frac{[-q^3][-q]}{[q^3][-q^4][-1]}$$
$$= q^{-1}\frac{[-q^2]}{[q][-1]} + q\frac{[-q^3]^2[q]}{[q^2][q^4][-q^4]^2} \qquad \textbf{3.0.11}$$

(To save time and space, I've written $[q^k]$ for $[q^k ; q^9]_\infty$). **3.0.9 - 11** arise from the zeroth, first and second components of $P^{-1} \times$ **3.0.4**, respectively, with the exponents of q divided by 3.

3.0.9, **3.0.10** and **3.0.11** may be proved by seven applications of **1.1.7**, with $n = 3$ and q^9 for q in each case. **3.0.9** needs just one application, with

$$(a_1, a_2, a_3 \,;\, b_1, b_2, b_3) = (1, q^3, q^3 \,;\, -1, -q^{-2}, q^8)$$

To prove **3.0.10**, I use **1.1.7** three times:

with $(a_1, a_2, a_3 \,;\, b_1, b_2, b_3) = (-q^2, -q^5, q^5 \,;\, 1, -q^2, q^{10})$, **1.1.7** gives

$$\frac{[-q^2][-q^3]}{[q^3][-q^4][-1]} = -q\,\frac{[-q^3]^2}{[q^4][-q^4]^2} + \frac{[-q][q^2][-q^3]}{[q][q^3][-q^4][-1]} \qquad \textbf{3.0.12}$$

and with $(a_1, a_2, a_3 \,;\, b_1, b_2, b_3) = (q, -q^2, q^3 \,;\, 1, -1, q^6)$ we have

$$\frac{[-q^2][-q^3]}{[q^3][-q^4][-1]} = -\frac{[-q][q^2][-q^3]}{[q][q^3][-q^4][-1]} + \frac{[q^4]}{[q][q^3]} \qquad \textbf{3.0.13}$$

while $(a_1, a_2, a_3 \,;\, b_1, b_2, b_3) = (-q^3, -q^4, q^{-1} \,;\, 1, q^4, q^2)$ provides

$$q^{-1}\,\frac{[-q^3][-q^4]}{[q^3][-q][-1]} = q^{-1}\,\frac{[-q^2][q^4]}{[q][q^2][-1]} - \frac{[q^4]}{[q][q^3]} \qquad \textbf{3.0.14}$$

Adding **3.0.12**, **3.0.13** and **3.0.14** gives **3.0.10**. **3.0.11** may also be proved using **1.1.7** three times, namely with $(a_1, a_2, a_3 \,;\, b_1, b_2, b_3) = (-q, -q^2, q^4 \,;\, 1, -q, q^6)$, $(-q, q^2, q^3 \,;\, 1, -1, q^6)$ and $(-q^2, q^3, q^7 \,;\, 1, q^2, -q^{10})$. qed

3.1 Here, I prove the "$\overset{7}{=}$"'s of theorem E, which I restate

Theorem

$$N(4,9,3n) = M(4,9,3n), \tag{3.1.0}$$

$$N(3,9,3n+1) + N(4,9,3n+1) = M(2,9,3n+1) + M(3,9,3n+1), \tag{3.1.1}$$

$$N(0,9,3n+2) = M(3,9,3n+2). \tag{3.1.2}$$

We have

$$\sum_{m,n} M(m,n) z^m q^n = \frac{(1-z)(q)_\infty}{[z;q]_\infty} \tag{3.1.3}$$

(since $M(m,n) = N_V(m,n)$ and, as I remarked just below **1.2.3**, the right hand-side of **3.1.3** is plainly the generating function of N_V). Setting $z = \exp(2\pi i/3)$ in **3.1.3**, we find that

$$M_3(0) - M_3(1) = \frac{(q)_\infty^2}{(q^3)_\infty}. \tag{3.1.4}$$

Defining $E := (q)_\infty^2/(q^3)_\infty$, **3.1.4** gives

$$M_9(0) + 2M_9(3) - M_9(1) - M_9(2) - M_9(4) = E,$$

which, together with the obvious

$$M_9(0) + 2M_9(1) + 2M_9(2) + 2M_9(3) + 2M_9(4) = P,$$

and the (known) "$\overset{4}{=}$"'s of theorem E, gives

$$9M_9(4)_0^{(3)} = P_0^{(3)} - E_0^{(3)}$$

$$9\big(M_9(2) + M_9(3)\big)_1^{(3)} = 2P_1^{(3)} + E_1^{(3)}$$

$$9M_9(3)_2^{(3)} = P_2^{(3)} + 2E_2^{(3)} \tag{3.1.5}$$

Now define functions $\mathcal{G}'_k = \mathcal{G}'_k(\tau)$ for k= 0, 1, 2 by

$$\exp(\pi i\tau/12)\mathcal{G}'_0(3\tau) = -9\big(PS_N(3,9)\big)^{(3)}_0 + E^{(3)}_0 - P^{(3)}_0,$$
$$\exp(\pi i\tau/12)\mathcal{G}'_1(3\tau) = 9\big(PS_N(2,9)\big)^{(3)}_1 + E^{(3)}_1 + 2P^{(3)}_1,$$
$$\exp(\pi i\tau/12)\mathcal{G}'_2(3\tau) = -9\big(PS_N(0,9)\big)^{(3)}_2 + E^{(3)}_2 - 4P^{(3)}_2. \qquad \mathbf{3.1.6}$$

and define functions $\mathcal{G}_k = \mathcal{G}_k(\tau)$ for k = 0, 1, 2 by

$$\exp(\pi i\tau/12)\mathcal{G}_0(3\tau) = \big(P(2S_N(3,9) - S_N(2,9))\big)^{(3)}_0$$
$$\exp(\pi i\tau/12)\mathcal{G}_1(3\tau) = -\big(P(2S_N(2,9) - S_N(0,9))\big)^{(3)}_1$$
$$\exp(\pi i\tau/12)\mathcal{G}_2(3\tau) = \big(P(2S_N(0,9) + S(3,9) + 1)\big)^{(3)}_2 \qquad \mathbf{3.1.7}$$

Then it follows from **1.3.4, 3.1.5** and the definitions **3.1.6** and **3.1.7** that, to establish **3.1.0, 3.1.1** and **3.1.2,** we must show that the $\mathcal{G}'_k$ are each identically zero. And, once we show that the $\mathcal{G}_k$ are identically zero, we'll have another proof of **3.0.0, 3.0.1** and **3.0.2.**

Define funtions $\mathcal{P}^{(n)}_k$, for $0 \le k < n$ by

$$\mathcal{P}^{(n)}_k(n\tau) = \exp(-\pi i\tau/12)P^{(n)}_k.$$

In this section, I write $\mathcal{P}_k$ for $\mathcal{P}^{(3)}_k$. I also define functions $\mathcal{E}_k = \mathcal{E}_k(\tau)$ for k = 0, 1 and 2, by

$$\mathcal{E}_k(3\tau) := \exp(-\pi i\tau/12)E^{(3)}_k.$$

Then **1.4.4** (with **2.5.4**) gives

$$\mathcal{P}_k = \eta(3)\eta(1)^{-4}\big(s(7^{k+2})^2 - s(7^{k+1})s(7^k)\big), \qquad \mathbf{3.1.8}$$

for k = 0, 1 and 2, (here, as I shall do throughout this section, I've written s(k) and t(k) for $s_9(k)$ and $t_9(k)$) and from **1.4.3** we have

$$\mathcal{E}_k = \eta(1)^{-1}\left(s(7^{k+2})^2 + 2s(7^{k+1})s(7^k)\right) \qquad \textbf{3.1.9}$$

For n not a multiple of 3, set

$$\mathcal{B}(m,n) = \mathcal{B}(m,n\,;\tau) := \frac{t(3n)t(m-3n)\eta(9)^3}{s(3n)t(m)t(0)},$$

and, for $n \equiv 0 \bmod 3$,

$$\mathcal{B}(m,n) = (-)^n\left\{-\frac{1}{3}\mathcal{Y}(m) - \frac{1}{9}\mathcal{Y}(3m)\right\},$$

where

$$\mathcal{Y}(m) = \mathcal{Y}(m\,;\tau) := \frac{s(2m)^3\eta(9)^3}{t(m)^3t(3m)}.$$

So $\mathcal{B}(m,n\,;3\tau) = q^{n(3n+1)/2+(4-m)n}B(m,n)$.

Note that, from **2.5.4** and **2.5.5**,

$$-\mathcal{Y}(-m) = \mathcal{Y}(m) = \mathcal{Y}(m+9),$$

$$-\mathcal{B}(-m,-n) = \mathcal{B}(m,n) = -\mathcal{B}(m,n+3) = \mathcal{B}(m+9,n) \qquad \textbf{3.1.10}$$

With these definitions and referring to **1.6.3** and **3.1.13** we have

$$\mathcal{G}'_k = -9\sum_{r \bmod 3}(-)^r\mathcal{P}_{k+r}\mathcal{B}(7^k, r7^k) + \mathcal{E}_k \qquad \textbf{3.1.11}$$

and

$$\mathcal{G}_k = \sum_{r \bmod 3}(-)^r\mathcal{P}_{k+r}\left(2\mathcal{B}(7^k, r7^k) + \mathcal{B}(7^{k+1}, r7^k)\right), \qquad \textbf{3.1.12}$$

for k = 0, 1 and 2.

I now show how $\Gamma_o(18)$ acts on the functions appearing on the right of **3.1.11** and **3.1.12**. First note that it follows from **2.4.1** that, for $A \in \Gamma_o(18)$,

$$\varepsilon(A^{(3)}) = \chi_6(a)\varepsilon(A)^3 \text{ and } \varepsilon(A^{(9)})^3 = \varepsilon(A)^3 \qquad \textbf{3.1.13}$$

(χ_6 is defined at **2.5.16**).

For n prime to 6, define $\rho(n)$ (modulo 3) by

$$7^{\rho(n)} \equiv \pm n \bmod 18.$$

Then **2.5.0**, **2.5.10**, **2.5.12** and **3.1.13** give

$$(\mathcal{P}_k|_{-1}A) = \sigma(\mathcal{P}_k; A)\mathcal{P}_{k+\rho(a)},$$
$$(\mathcal{E}_k|_1 A) = \sigma(\mathcal{E}_k; A)\mathcal{E}_{k+\rho(a)}$$

(the subscript $k+\rho(a)$ being reckoned mod 3, remember that a is the top left entry of A) and

$$\left(\mathcal{B}(m,n)|_2 A\right) = \sigma\left(\mathcal{B}(m,n); A\right)\mathcal{B}(am, an)$$

where

$$\sigma(\mathcal{P}_k; A) = \chi_6(a)\exp(-4^{k+1}\pi iab/9)\varepsilon(A)^6,$$
$$\sigma(\mathcal{E}_k; A) = \exp(-4^{k+1}\pi iab/9)\varepsilon(A)^4,$$
$$\sigma\left(\mathcal{B}(m,n); A\right) = \exp(-2mnab/3)\varepsilon(A)^{-2}. \qquad \textbf{3.1.14}$$

So, if X is any one of the summands on the RHS of **3.1.12** and $A \in \Gamma_o(18)$,

$$\sigma(X; A) = \chi_6(a)\exp(-4^{k+1}\pi iab/9)\varepsilon(A)^4$$

(this follows from the elementary fact that $4^r \equiv 3r+1 \bmod 9$). Since these theta products all have the same (negative) sign, they are coherent. The weight of each is 1/2 and it follows that each $\mathcal{G}_k$ is a modular form of weight 1/2 on $\Gamma_1(18)$.

It also follows from **3.1.14** that, for $A_7 := \begin{pmatrix} 7 & -2 \\ 18 & -5 \end{pmatrix} \in \Gamma_0(18)$ and for s = 0, 1 or 2,

$$(\mathcal{G}_0 |_1 A_7^s) = \exp(8\pi i s/9)\varepsilon(A_7)^{4s}\mathcal{G}_s \qquad \textbf{3.1.15}$$

So, if any one of the $\mathcal{G}_k$ is identically zero, so are the other two.

Likewise, we find that the summands in **3.1.11** are coherent and we have

$$(\mathcal{G}'_k |_1 A) = \upsilon(\mathcal{G}'_k ; A)\mathcal{G}'_{k+\rho(a)}$$

with

$$\upsilon(\mathcal{G}'_k ; A) = \exp(-4^{k+1}\pi i ab/9)\varepsilon(A)^4.$$

So each of the $\mathcal{G}'_r$ is a modular form on $\Gamma_1(18)$ and, if any one of the $\mathcal{G}'_r$ is identically zero, so are the other two.

2.1.5 gives the index of $\Gamma_1(18)$ in $SL_2(\mathbb{Z})$. It is

$$\mu = 1/2 \times 18 \times 18 \times 3/4 \times 8/9 = 108.$$

Now $\mathcal{G}'_0$ is a modular form of weight 1/2 on $\Gamma_1(18)$, so, by **2.4.7**, to show that $\mathcal{G}'_0$ is identically zero, it is enough to show that

$$\sum_{\zeta \in C^{(18)}} \mathrm{ord}(\mathcal{G}'_0, \zeta, \Gamma_1(18)) > 108 \times 1/2 \times 1/12 = 9/2. \qquad \textbf{3.1.16}$$

The table **A0.2** shows that each $\mathcal{B}(m,n)$ has nonnegative order at each cusp in $C' := C^{(18)} \setminus C^{(18)}_{18}$, and the lower bounds for the orders of the $\mathcal{P}_k$ given in this table are also lower bounds for the orders of the $\mathcal{E}_k$, so we have

$$\sum_{\zeta \in C'} \mathrm{ord}(\mathcal{G}'_0, \zeta, \Gamma_1(18)) \geq 3\times -9/4 + 3\times -9/8 + 2\times -1/12 + 2\times -1/24 + 3\times -1/36$$
$$= -251/24$$

By **3.1.16**, it is ∴ enough to show that

$$\mathrm{ord}(\mathcal{G}'_0, 1/18, \Gamma_1(18)) + \mathrm{ord}(\mathcal{G}'_0, 5/18, \Gamma_1(18)) + \mathrm{ord}(\mathcal{G}'_0, 7/18, \Gamma_1(18))$$
$$> 9/2 + 251/24 = 15 - 1/24 \qquad \textbf{3.1.17}$$

Now

$$\mathrm{ord}(\mathcal{G}'_0, 1/18, \Gamma_1(18) = \mathrm{ord}\, \mathcal{G}'_0$$

and, by **3.1.15**,

$$\mathrm{ord}(\mathcal{G}'_0, 5/18, \Gamma_1(18)) = \mathrm{ord}(\mathcal{G}'_2, 1/18, \Gamma_1(18)) = \mathrm{ord}\, \mathcal{G}'_2$$
$$\mathrm{ord}(\mathcal{G}'_0, 7/18, \Gamma_1(18)) = \mathrm{ord}(\mathcal{G}'_1, 1/18, \Gamma_1(18)) = \mathrm{ord}\, \mathcal{G}'_1.$$

From their definitions,

$$\mathcal{G}'_0 = q^{-1/72} \times 9\sum_{n\geq 0}\big(N(4,9,3n) - M(4,9,3n)\big)q^n$$
$$\mathcal{G}'_1 = -q^{-1/72} \times 9\sum_{n\geq 0}\big(N(3,9,3n+1) + N(4,9,3n+1)$$
$$- M(2,9,3n+1) - M(3,9,3n+1)\big)q^{n+1/3}$$
$$\mathcal{G}'_2 = -q^{-1/72} \times \frac{9}{2}\sum_{n\geq 0}\big(N(0,9,3n) - M(3,9,3n)\big)q^{n+2/3},$$

so, once we've verified the first five cases of each of **3.1.0**, **3.1.1** and **3.1.2**, we will have

$$\mathrm{ord}(\mathcal{G}'_0, 1/18, \Gamma_1(18)) + \mathrm{ord}(\mathcal{G}'_0, 1/18, \Gamma_1(18)) + \mathrm{ord}(\mathcal{G}'_0, 1/18, \Gamma_1(18))$$
$$\geq 5 - 1/72 + 5 + 1/3 - 1/72 + 5 + 2/3 - 1/72$$
$$= 16 - 1/24 > 15 - 1/24.$$

and **3.1.17** will tell us that, to show that **3.1.0**, **3.1.1** and **3.1.2** are always true.

Exactly the same argument serves to show that, to demonstrate the general truth of **3.0.0**, **3.0.1** and **3.0.2**, it is enough also to verify the first five cases of each of them. The data needed to make these verifications are provided in table **A.1.0**.

In fact, a closer look at $\mathcal{G}_0$ (and appealing to **1.2.16**) shows that, to prove **3.0.0**, **3.0.1** and **3.0.2**, it is enough to check the first <u>three</u> cases of each of them. (This number seems remarkably low.)

3.2 In this section, I prove the "$\overset{5}{=}$"'s and "$\overset{*}{=}$"'s of theorem D. I restate them as the following three theorems.

Theorem

$$N(2,8,4n) = N(4,8,4n), \qquad \mathbf{3.2.0}$$

$$N(1,8,4n+1) + N(2,8,4n+1) = N(3,8,4n+1) + N(4,8,4n+1), \qquad \mathbf{3.2.1}$$

$$N(0,8,4n+2) = N(2,8,4n+2), \qquad \mathbf{3.2.2}$$

$$N(0,8,4n+3) + N(1,8,4n+3) = N(2,8,4n+3) + N(3,8,4n+3). \qquad \mathbf{3.2.3}$$

Theorem

$$N(2,8,4n) = M(3,8,4n), \qquad \mathbf{3.2.0'}$$

$$N(1,8,4n+1) + N(2,8,4n+1) + N(3,8,4n+1) = M(0,8,4n+1) + M(1,8,4n+1) + M(2,8,4n+1), \qquad \mathbf{3.2.1'}$$

$$N(2,8,4n+2) = M(1,8,4n+2), \qquad \mathbf{3.2.2'}$$

$$N(1,8,4n+3) + N(2,8,4n+3) + N(3,8,4n+3)$$
$$= M(2,8,4n+3) + M(3,8,4n+3) + M(4,8,4n+3), \qquad \mathbf{3.2.3'}$$

Theorem

$$N(3,8,4n) = M(2,8,4n), \qquad \mathbf{3.2.0''}$$
$$N(0,8,4n+1) + N(1,8,4n+1) + N(2,8,4n+1)$$
$$= M(1,8,4n+1) + M(2,8,4n+1) + M(3,8,4n+1), \qquad \mathbf{3.2.1''}$$
$$N(1,8,4n+2) = M(2,8,4n+2), \qquad \mathbf{3.2.2''}$$
$$N(2,8,4n+3) + N(3,8,4n+3) + N(4,8,4n+3)$$
$$= M(1,8,4n+3) + M(2,8,4n+3) + M(3,8,4n+3), \qquad \mathbf{3.2.3''}$$

Note that **3.2.1′**, **3.2.3′**, **3.2.1′′** and **3.2.3′′** do not explicitly appear in theorem D and that some of the "$\overset{5}{=}$"s of theorem D do not appear here. However, it is easy to see that statements of theorem D are equivalent to **3.2.0-3′′** together with the (known) "$\overset{4}{=}$"s. (The "$\overset{4}{=}$"s have elementary proofs, see **[Gar2]**. They may also be proved by the methods I use here.)

For example, writing N(r) and M(r) for N(r, 8, 4n + 1) and M(r, 8, 4n + 1), **3.2.1′**, **3.2.1′′** and the trivial

$$N(0) + 2N(1) + 2N(2) + 2N(3) + N(4)$$
$$= M(0) + 2M(1) + 2M(2) + 2M(3) + M(4)$$

give

$$N(3) + N(4) = M(3) + M(4), \qquad \mathbf{3.2.4}$$

which is one of the two missing "$\overset{5}{=}$"'s of the $4n+1$ case of theorem D. Now

$$\begin{aligned} M(2) + N(3) + N(4) &= M(2) + M(3) + M(4) && \text{(by } \mathbf{3.2.4}) \\ &= M(0) + M(1) + M(2) && \text{(by one of the "}\overset{4}{=}\text{"'s)} \\ &= N(1) + N(2) + N(3) && \text{(by } \mathbf{3.2.1'}) \\ &= 2N(3) + N(4) && \text{(by } \mathbf{3.2.1}) \end{aligned}$$

which gives $N(3) = M(2)$, the other missing "$\overset{5}{=}$" of the $4n+1$ case.

(Here's an amusing way of viewing **3.2.0-3**. Set

$$Q := 2P \sum (-)^n \frac{q^{n(3n+1)/2}}{1+q^{2n}} ,$$

$$R := 2P \sum (-)^n \frac{q^{n(3n+1)/2}}{1+q^{4n}} ,$$

$$S := 2P \sum (-)^n \frac{q^{3n(n+1)/2}}{1+q^{4n}} .$$

It can be seen from **0.2.4** and the methods of §**1.3** that

$$Q = N_8(0) + N_8(4) - 2N_8(2).$$

$$R = N_8(0) - N_8(4).$$

$$S = N_8(0) + 2N_8(1) - 2N_8(3) - N_8(4).$$

So **3.2.0**, **3.0.1**, **3.0.2** and **3.0.3** may be read as

$$Q_0^{(4)} = R_0^{(4)},\ Q_1^{(4)} = S_1^{(4)},\ Q_2^{(4)} = -R_2^{(4)},\ Q_3^{(4)} = -S_3^{(4)}.)$$

In this section, I write $s(k)$, $t(k)$ and $\mathcal{P}_k$ for $s_{16}(k)$, $t_{16}(k)$ and $\mathcal{P}_k^{(4)}$. **1.4.5** gives

$$\mathcal{P}_k = t(3^{k+2})\mathcal{K} + t(3^k)\mathcal{L}, \qquad \mathbf{3.2.5}$$

where

$$\mathcal{K} = \eta(1)^{-5}\eta(4)\eta(16)^{-3}s(4)^4 s(8)$$

$$\text{and } \mathcal{L} = 2\eta(1)^{-4}\eta(16)^{-3}s(8)t(1)t(3)t(5)t(7). \qquad \textbf{3.2.6}$$

For integers m and n with $m \not\equiv \pm n \bmod 16$, define

$$\mathcal{A}_N(m,n) = \mathcal{A}_N(m,n;\tau) := \frac{t(n)s(2n)\eta(16)^3}{s(m-n)t(m)s(m+n)} \quad \text{and}$$

$$\mathcal{A}_M(m,n) = \mathcal{A}_M(m,n;\tau) := \frac{t(m)\,s(2n)\eta(16)^3}{s(m-n)t(n)s(m+n)}$$

and, for $m \not\equiv 0 \bmod 16$,

$$\mathcal{H}(m) := \frac{s(2m)^3\eta(16)^3}{t(m)^3 t(3m)}, \qquad \mathcal{H}'(m) := \frac{s(4m)^3\eta(16)^3}{s(2m)^3 s(6m)}$$

and then, for $m \equiv \pm n \not\equiv 0 \bmod 16$,

$$\mathcal{A}_N(m,n) := -\frac{1}{80}\sum_{r=0}^{3} 3^r\left(\mathcal{H}'(3^{3-r}n) + \mathcal{H}(3^{3-r}n)\right)$$

$$\mathcal{A}_M(m,n) := -\frac{1}{80}\sum_{r=0}^{3} 3^r\left(\mathcal{H}'(3^{3-r}n) - \mathcal{H}(3^{3-r})\right).$$

Comparing these definitions with those in §1.5, we have

$$\mathcal{A}_N(m,n;4\tau) = q^{3(n-m)(n+m-16)/8}\, A_N(4m,4n),$$

$$\mathcal{A}_M(m,n;4\tau) = q^{(n-m)(n+m-16)/8} A_M(4m,4n),$$

Note that, for all $m, n \not\equiv 0 \bmod 16$ and $* = M$ or N,

$$\mathcal{A}_*(m+16,n) = \mathcal{A}_*(-m,n) = \mathcal{A}_*(m,n) = -\mathcal{A}_*(m,-n) = \mathcal{A}_*(m,n+16) \qquad \textbf{3.2.7}$$

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(16)$ and set $c' := c/16$. Then it is easy to see (using **2.5.0**) that

$$\varepsilon(A^{(4)}) = (-)^{c'}\exp(\pi i ab/4)\varepsilon(A) \text{ and } \varepsilon(A^{(16)}) = \exp\big(5\pi i a(b+c')/4\big)\varepsilon(A). \qquad \mathbf{3.2.8}$$

We also have from **2.5.10**, **2.5.12** and **3.2.8**

$$o\big(s_n(k)\,;A\big) = (-)^{kb}\exp(\pi i k^2 ab/16)\exp\big(-\pi i a(b+c')/4\big)\varepsilon(A)^2,$$

$$o\big(t_n(k)\,,A\big) = (-)^{kb}i^{1-d}\exp(\pi i k^2 ab/16)\exp(-\pi i ab/4)\varepsilon(A)^2 \qquad \mathbf{3.2.9}$$

(using the fact that, for $A \in \Gamma_0(16)$, $a \equiv d \bmod 8$). From **3.2.8** and **3.2.9** and referring to the definitions **3.2.6**, we have

$$o(\mathcal{K}\,;A) = (-)^b\exp(\pi i ab/4)\varepsilon(A)^{10} \text{ and } o(\mathcal{L}\,;A) = \exp(\pi i ab/4)\varepsilon(A)^{10}$$

and so the two theta products in each of the $\mathcal{P}_r$ at **3.2.5**. are compatible. If A is a matrix in $\Gamma_0(32)$ and A' a matrix in $\Gamma_0(16)\backslash\Gamma_0(32)$ (i.e. in $\Gamma_0(16)$ but not in $\Gamma_0(32)$), we have

$$\big(\mathcal{P}_k\,|_{-1}A\big) = o\big(\mathcal{P}_k\,;A\big)\mathcal{P}_{k+\rho(a)}, \quad \big(\mathcal{P}_k\,|_{-1}A'\big) = o\big(\mathcal{P}_k\,;A'\big)\mathcal{P}_{k+\rho(a)} \qquad \mathbf{3.2.10}$$

where now $\rho(n)$ is defined (modulo 4) for odd n by

$$n \equiv \pm 3^{\rho(n)} \bmod 16, \qquad \mathbf{3.2.11}$$

the subscripts $\alpha+k$ and $\alpha+k+2$ are evaluated mod 4 and

$$o\big(\mathcal{P}_k\,;A\big) = o\big(\mathcal{P}_k\,;A'\big) = \exp(\pi i 3^{2k}ab/16) \qquad \mathbf{3.2.12}$$

We also have from **2.5.0**, **2.5.10** and **2.5.11**

$$o\big(\mathcal{H}(m)\,;A\big) = \varepsilon(A)^{-2} = o\big(\mathcal{H}'(m)\,;A\big)$$

so the theta products comprising $\mathcal{A}_*(m,n)$ are compatible when $m \equiv \pm n \bmod 16$ (for $* = N$ or M). We find that

$$\upsilon\left(\mathcal{A}_N(m\,,n)\,;A\right) = \exp\left(\pi i 3(n^2 - m^2)ab/16\right)\varepsilon(A)^{-2}$$

$$\upsilon\left(\mathcal{A}_M(m\,,n)\,;A\right) = \exp\left(\pi i(n^2 - m^2)ab/16\right)\varepsilon(A)^{-2} \qquad \mathbf{3.2.13}$$

For k = 0, 1, 2 and 3, define the functions

$$\mathcal{F}_k := \sum_{s=0}^{3}(-)^s\mathcal{P}_{k+s}\Big(\mathcal{A}_N(3^k, 3^{k+s}) - \mathcal{A}_N(7.3^k, 7.3^{k+s}) - 2\mathcal{A}_N(5.3^k, 5.3^{k+s})\Big)$$

$$\mathcal{F}'_k := \sum_{s=0}^{3}(-)^s\mathcal{P}_{k+s}\Big(\mathcal{A}_N(3^k, 3^{k+s}) - \mathcal{A}_N(7.3^k, 7.3^{k+s}) - \mathcal{A}_M(3^k, 3^{k-s}) + \mathcal{A}_M(3.3^k, 3.3^{k-s})\Big)$$

$$\mathcal{F}''_k := \sum_{s=0}^{3}(-)^s\mathcal{P}_{k+s}\Big(\mathcal{A}_N(3^k, 3^{k+s}) + \mathcal{A}_N(5.3^k, 5.3^{k+s}) + \mathcal{A}_M(3.3^k, 3.3^{k-s}) - \mathcal{A}_M(5.3^k, 5.3^{k-s})\Big)$$

in which $k+s$ and $k-s$ are to be interpreted as their least non-negative residues modulo 4. Then careful inspection of **1.5.11** and **3.2.7** shows that

$$\mathcal{F}_0 = \exp(-\pi i\tau/48)\sum\Big(N(2\,,8\,,4n) - N(4\,,8\,,4n)\Big)q^n,$$

$$\mathcal{F}_1 = \exp(11\pi i\tau/48)\sum\Big(N(1\,,8\,,4n+1) + N(2\,,8\,,4n+1) - N(3\,,8\,,4n+1) - N(4\,,8\,,4n+1)\Big)q^n,$$

$$\mathcal{F}_2 = \exp(23\pi i\tau/48)\sum\Big(N(2\,,8\,,4n+2) - N(0\,,8\,,4n+2)\Big)q^n,$$

$$\mathcal{F}_3 = \exp(35\pi i\tau/48)\sum\Big(N(2\,,8\,,4n+1) + N(3\,,8\,,4n+1) - N(0\,,8\,,4n+1) - N(1\,,8\,,4n+1)\Big)q^n.$$

So, **3.2.k** is equivalent to $\mathcal{F}_k \equiv 0$ (k = 0, 1, 2, 3). And $\mathcal{F}'_k$ and $\mathcal{F}''_k$ are, to the same extent, the generating functions of the difference between the two sides of **3.2.k′** and **3.2.k″**.

Now it is not difficult to see from **3.2.13** that, for odd u and v and $A \in \Gamma_0(16)$,

$$o\big(\mathcal{A}_N(u.3^k, u.3^{k+s}); A\big) = \exp\big((3^{2k} - 3^{2k+2s})\pi i ab/16\big)$$
$$= o\big(\mathcal{A}_M(v.3^k, v.3^{k-s}); A\big),$$

so the various theta products making up each $\mathcal{F}_k$, $\mathcal{F}'_k$ or $\mathcal{F}''_k$ are compatible.

From **3.2.10** and **3.2.11** we have, for $\mathcal{F}^*_k$ any one of $\mathcal{F}_k$, $\mathcal{F}'_k$ and $\mathcal{F}''_k$,

$$\big(\mathcal{F}^*_k|_1 A\big) = o\big(\mathcal{F}^*_k; A\big)\mathcal{F}^*_{\alpha+k}, \quad \big(\mathcal{F}^*_k|_1 A'\big) = o\big(\mathcal{F}^*_k; A'\big)\mathcal{F}^*_{\alpha+k+2} \qquad \textbf{3.2.14}$$

where A, A′ lie in $\Gamma_0(32)$, respectively $\Gamma_0(16)\backslash\Gamma_0(32)$, α is as at **3.2.11**, and

$$o\big(\mathcal{F}^*_k; A\big) = \big(\mathcal{F}^*_k; A'\big) = \exp(\pi i 3^{2k} ab/16).$$

It follows from **3.2.14** that, if any one of the $\mathcal{F}^*_k$ is identically zero, so are the other three. Now $\Gamma_1(32)$ has index

$$\frac{1}{2} \times 32 \times 32 \times \frac{3}{4} = 384$$

in $SL_2(\mathbb{Z})$, by **2.1.5**, and, by **3.2.14**, (each of the) $\mathcal{F}^*_0$ is a modular form of weight 1/2 on $\Gamma_1(32)$. Also, each $\mathcal{F}^*_0$ is holomorphic on $\mathbb{H}$. So, to show that $\mathcal{F}^*_0$ is identically zero, it is enough, by **2.4.7**, to show that

$$\sum_{\zeta \in \mathcal{C}^{(32)}} \operatorname{ord}\big(\mathcal{F}^*_0, \zeta, \Gamma_0(32)\big) > 384/24 = 16. \qquad \textbf{3.2.15}$$

Suppose a/32 is one of the eight cusps in $\mathcal{C}^{(32)}_{32}$. Then **3.2.14** shows that

$$\operatorname{ord}\big(\mathcal{F}^*_0, a/32, \Gamma_0(32)\big) = \operatorname{ord}\mathcal{F}_{\rho(a)}$$

and, if a/16 is any one of the four cusps in $\mathcal{C}^{(32)}_{16}$, then

$$\operatorname{ord}\big(\mathcal{F}^*_0, a/16, \Gamma_0(32)\big) = 2 \times \operatorname{ord}\mathcal{F}_{\rho(a)+2}$$

(since cusps in $C_{16}^{(32)}$ have width 2). Now the table **A.1.1** shows that the first four cases of each **3.2.k*** hold, so

$$\operatorname{ord}\mathcal{F}_k^* \geq 4 - 1/96 + k/4$$

and $\therefore$

$$\sum_{\zeta \in C''} \operatorname{ord}\left(\mathcal{F}_0^*, \zeta, \Gamma_0(32)\right) \geq 64 - 1/6 + 6 > 60 - 1/6, \qquad \textbf{3.2.16}$$

where $C'' := C_{16}^{(32)} \cup C_{32}^{(32)}$ and (below) $C' := C^{(32)} \backslash C''$. Table **A.0.3** shows that each of the $\mathcal{A}_N(m,n)$ and $\mathcal{A}_M(m,n)$ has non-negative order at each cusp in C' (this is not true at all cusps, e.g. $\mathcal{A}_N(1,3)$ has order -1/2 at 7/16) and gives lower bounds for the orders of the $\mathcal{P}_k$ at the cusps in C'. Since

$$|C_1^{(32)}| = 8 \text{ and } |C_2^{(32)}| = |C_4^{(32)}| = |C_8^{(32)}| = 4,$$

we have

$$\sum_{\zeta \in C'} \operatorname{ord}\left(\mathcal{F}_0^*, \zeta, \Gamma_0(32)\right) \geq 8 \times -16/3 + 4 \times -1/6 + 4 \times -1/12 + 4 \times -1/24$$
$$= -44 + 1/6 \qquad \textbf{3.2.17}$$

3.2.17 and **3.2.16** give **3.2.15.** (In fact, it is enough to check the first four cases of two of the $\mathcal{F}^*$ and the first three cases of the other two.)

3.3 The last of the main theorems is theorem H, namely

Theorem

$$N(2, 12, 2n) = N(5, 12, 2n), \qquad \textbf{3.3.0}$$

$$N(4, 12, 2n+1) = N(1, 12, 2n+1). \qquad \textbf{3.3.1}$$

for k = 0, 1, and we have

$$\left(\mathcal{P}_k|_{-1}A\right) = o(\mathcal{P}_k;A)\mathcal{P}_k, \text{ if } a \equiv \pm 1 \text{ or } \pm 7 \bmod 24,$$

$$= o(\mathcal{P}_k;A)\mathcal{P}_{1-k}, \text{ if } a \equiv \pm 5 \text{ or } \pm 11 \bmod 24.$$

We also have

$$o(\mathcal{Q}(m);A) = \left(\tfrac{6}{|d|}\right)\exp\left(-\pi i(c' + b)a/4\right)\varepsilon(A)^2,$$

which is independent of m, and so the three summands making up $C(m,0)$ are compatible. We find that, for odd m,

$$o\left(C(m,n);A\right) = \left(\tfrac{6}{|d|}\right)(-)^{bn(n+m)/2}\exp\left(-\pi i(c' + b)a/4\right)\varepsilon(A)^2.$$

Define the functions

$$\begin{aligned}\mathcal{H}_0 := \mathcal{P}_0\big(&C(7,0) - C(7,1) - C(9,0) + C(9,3)\\ &- C(1,0) + C(1,3) + C(3,0) - C(3,1)\big)\\ + \mathcal{P}_1\big(&C(7,2) - C(7,3) - C(9,2) + C(9,1)\\ &- C(1,2) + C(1,1) + C(3,2) - C(3,3)\big)\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_1 := \mathcal{P}_1\big(&C(3,0) - C(3,1) - C(5,0) + C(5,3)\\ &- C(9,0) + C(9,3) + C(11,0) - C(11,1)\big)\\ + \mathcal{P}_0\big(&C(3,2) - C(3,3) - C(5,2) + C(5,1)\\ &- C(9,2) + C(9,1) + C(11,2) - C(11,3)\big)\end{aligned}$$

Then **1.7.6** shows that

$$\mathcal{H}_0 = \exp(-\pi i\tau/24)\sum\left(N(2,12,2n) - N(5,12,2n)\right)q^n,$$

$$\mathcal{H}_1 = \exp(23\pi i\tau/24)\sum\left(N(4,12,2n+1) - N(1,12,2n+1)\right)q^n$$

and **3.3.0** and **3.3.1** amount to

$$\mathcal{H}_0 \equiv 0, \mathcal{H}_1 \equiv 0.$$

It may be checked that, if $\mathcal{P}_r\mathcal{C}(m, n)$ $(r = 0, 1)$ is a summand of either of the $\mathcal{H}_k$, then

$$r + n(n+m)/2 \equiv k \bmod 2.$$

It follows that, in each of the $\mathcal{H}_k$, the summands are compatible and we have

$$\begin{aligned}(\mathcal{H}_k\,|_1\,A) &= o(\mathcal{H}_k\,;A)\mathcal{H}_k, \text{ if } a \equiv \pm 1 \text{ or } \pm 7 \bmod 24\\ &= o(\mathcal{H}_k\,;A)\mathcal{H}_{1-k}, \text{ otherwise,}\end{aligned} \qquad \mathbf{3.3.2}$$

where

$$o(\mathcal{H}_k\,;A) = (-)^{kb+(d-1)/2}\exp(3\pi iab/8)\exp\left(-\pi i(c'+b)a/4\right)\varepsilon(A)^{10}.$$

In particular, each $\mathcal{H}_k$ is a modular form of weight 1/2 on $\Gamma_1(48)$. Also, if either of the $\mathcal{H}_k$ is identically zero, so is the other.

Set $\mathcal{C}'' = \mathcal{C}^{(48)}_{48}$ and $\mathcal{C}' := \mathcal{C}^{(48)}\setminus\mathcal{C}''$. It may be checked that, for $r \neq 16$ or 48, each $\mathcal{C}(m, n)$ (with m odd) has nonnegative order at each cusp in $\mathcal{C}^{(48)}_r$, while at the cusps in $\mathcal{C}^{(48)}_{16}$, each $\mathcal{C}(m, n)$ has order at least -1/2. And the orders of each $\mathcal{P}_k$ at the cusps in $\mathcal{C}^{(48)}_r \subseteq \mathcal{C}'$ are:

-4/r, when r = 1 or 3, and -1/r, when r = 2, 4, 6, 8, 12, 16 or 24.

Since

$|\mathcal{C}^{(48)}_r| = 8$, when r = 1, 3 or 16, and = 4, when r = 2, 4, 6, 8, 12 or 24, we have

$$\sum_{\zeta \in C'} \mathrm{ord}\left(\mathcal{H}_0, \zeta, \Gamma_1(48)\right) \geq 8 \times -4 + 4 \times -1/2 + 8 \times -4/3 + 4 \times -1/4 + 4 \times -1/6$$

$$+ 4 \times -1/8 + 4 \times -1/12 + 8 \times -9/16 + 4 \times -1/24.$$

$$= -52 + 1/6. \qquad \mathbf{3.3.3}$$

We see from table **A.1.2** that $\mathcal{H}_0$ has order at least 11 - 1/48 and that $\mathcal{H}_1$ has order at least 11 + 23/48. So

$$\sum_{\zeta \in C''} \mathrm{ord}\left(\mathcal{H}_0, \zeta, \Gamma_1(48)\right) \geq 90 - 1/6 \qquad \mathbf{3.3.4}$$

and **3.3.3** and **3.3.4** give

$$\sum_{\zeta \in C(48)} \mathrm{ord}\left(\mathcal{H}_0, \zeta, \Gamma_1(48)\right) \geq 38 > 32. \qquad \mathbf{3.3.5}$$

Now it may be seen from **2.1.5** that $\Gamma_1(48)$ has index

$$\frac{1}{2} \times 48 \times 48 \times \frac{3}{4} \times \frac{8}{9} = 768$$

in $SL_2(\mathbb{Z})$ and

$$768/24 = 32.$$

Furthermore $\mathcal{H}_0$ is holomorphic in $\mathbb{H}$ and so **2.4.7** and **3.3.5** establish **3.3.0** and **3.3.1**.

3.4 Define

$$Q(i,j,k) := \sum_n \big(N(i,9,3n+k) - N(j,9,3n+k)\big)q^n,$$

$$R(i,j,k) := \sum_n \big(N(i,8,4n+k) - M(j,8,4n+k)\big)q^n,$$

$$V^*(m) := V(m;q^{1/3}) = [-1;q^9]_\infty^{-1}(q^9)_\infty^{-1}T_1(-q^m,-1,q^9),$$

$$U^*(m) := U(m;q^{1/4}) = (q^{16})_\infty^{-1}T_3(-q^m,1,q^{16}),$$

$$\prod\left({x,y,\ldots \atop u,v,\ldots};q^n\right) := \frac{[x;q^n]_\infty[y;q^n]_\infty\cdots}{[u;q^n]_\infty[v;q^n]_\infty\cdots}(q^n)_\infty .$$

In the appendix **B.0**, I list the generating functions of the $Q(i,0,k)$, for all $1 \le i \le 4$ and $0 \le k \le 2$ (and that of $Q(3,2,0)$), expressed in terms of the functions V^* and $\prod$. And in the appendix **B.1**, I list the generating functions, expressed in terms of U^* and $\prod$, of all the $R(i,j,k)$. The "$\underline{\underline{6}}$"'s of theorem E follow from **B.0** (**1.1.7** is needed for the $3n+1$ case), while all theorem D appears in **B.1**. Let us denote these "identities" by

$$Q(i,j,k) = S(i,j,k) \text{ and } R(i,j,k) = T(i,j,k)$$

To show that they all hold, one shows that each

$$q^{(24k-1)/72}\times\big(Q(i,j,k) - S(i,j,k)\big),$$

respectively

$$q^{(24k-1)/96}\times\big(R(i,j,k) - T(i,j,k)\big),$$

is a modular form of weight 1/2 on $\Gamma_1(18)$, respectively $\Gamma_1(32)$. Then, as in §§**3.1**, **3.2**, it only remains to show that the first so many cases of each identity hold. In the case of the identities **B.0**, examination of table **A.0.1** shows that, to prove any one of them, we need to examine no more than

$$9/2 + 3\times 9/4 + 3\times 9/8 + 2\times 1/12 + 2\times 1/24 + 3\times 1/36 + 2\times 49/72 + \varepsilon$$
$$= 1175/72 + \varepsilon$$

cases. So 17 cases will do. This is well within the scope of the double precision integer arithmetic available on my PC and I have checked each of the identities **B.0** to this extent.

In the case of the identities **B.1**, the same sum reveals that we have to check at least 6119/96 > 63 cases. Such a number is beyond the reach of my PC (it runs out of steam, proclaiming "number too large", after 32 cases). So we need a more sophisticated computer (or a more sophisticated program) or the following observation. Set

$$A := \begin{pmatrix} 15 & 7 \\ 32 & 15 \end{pmatrix},\ B := \begin{pmatrix} 1 & 1 \\ 16 & 17 \end{pmatrix},\ C := \begin{pmatrix} 7 & 3 \\ 16 & 7 \end{pmatrix},\ D := \begin{pmatrix} 7 & -2 \\ 32 & -9 \end{pmatrix} \text{ and } E := \begin{pmatrix} 9 & -2 \\ 32 & -7 \end{pmatrix}$$

and

$$\mathcal{D}(i,j,k) := q^{(24k-1)/96} \times \big(R(i,j,k) - T(i,j,k)\big).$$

Then we have

$$\big(\mathcal{D}(i,j,k)\,|_1 A\big) = u\mathcal{D}(i,j,k) = \big(\mathcal{D}(i,j,k)\,|_1 C\big)$$
$$\big(\mathcal{D}(i,j,k)\,|_1 B\big) = u\mathcal{D}(4-i,4-j,k+2) = \big(\mathcal{D}(i,j,k)\,|_1 D\big) = \big(\mathcal{D}(i,j,k)\,|_1 E\big)$$

where u is a root of unity (not necessarily the same for each equality) and the number $k+2$ is evaluated mod 4. Following the argument of §**3.2**, once we have checked the first eight cases of each of

$$R(i,j,k) = T(i,j,k) \text{ and } R(4-i,4-j,k+2) = T(4-i,4-j,k+2)$$

we'll have

$$\sum_{\zeta \in \mathcal{C}^{(32)}} \mathrm{ord}\big(\mathcal{D}(i,j,k), \zeta, \Gamma_1(32)\big) \geq 8 \times -16/3 + 4 \times -1/6 + 4 \times -1/12 + 4 \times -1/24$$

$$+ 2 \times -25/48 + 4 \times -25/96 + 4 \times (8 - 1/96) + 4 \times (8 + 47/96)$$

$$> 16 = [SL_2(\mathbf{Z}) : \Gamma_1(32)]/24$$

and $\mathcal{D}(i,j,k)$ is identically zero. I have checked out the first eight (indeed, the first 30) cases of each of the identities **B.1**, so each is true.

I am fairly certain that the identities **B.0** and **B.1** could also be proved by using **1.1.7** in the manner of **§3.0**.

Define the ***deficit*** of the theta product $\mathcal{X}(\mathbf{m})$ at **2.5.13** to be the number

$$\mathrm{len}\,\mathbf{u} + \mathrm{len}\,\mathbf{v} - \mathrm{len}\,\mathbf{x} - \mathrm{len}\,\mathbf{y}.$$

(I'm not at all sure how sound this definition is. The same function may appear as a theta product of index n in different ways with different deficits, at least when n is even. e.g.

$$s_{2n}(2m)\eta(2n)^3 = s_{2n}(m)s_{2n}(n-m)t_{2n}(m)t_{2n}(n-m).$$

What we need is a canonical form for theta products, if such exists.) Then each of the theta products appearing on the right hand side of the identities **B.0** have deficit -2, while those appearing on the right hand side of the **B.1**'s each have deficit -1. I have no explanation of these phenomena (or, at best, a vague, unformed idea).

3.5 A couple of trivial remarks. **1.0.1** gives

$$[-q^a ; q^n]_\infty (q^n)_\infty = \sum_n q^{an} q^{n(n-1)/2},$$

and so the product on the left has no negative coefficients. The same is true of

$$[q^a ; q^n]_\infty^{-1} (q^n)_\infty = [q^{2a} ; q^{2n}]_\infty^{-1} [-q^a ; q^n]_\infty (q^n)_\infty ,$$

for $0 < a < n$. It follows that a power series

$$\prod \left(\begin{matrix} -q^a, -q^b, -q^c, \ldots \\ q^r, \ q^s, \ q^t, \ldots \end{matrix} ; q^n \right),$$

with at least one term before the semicolon (and $0 < r, s, t, \ldots < n$), has all its coefficients nonnegative. With this (and other trivial considerations) in mind, the identities **B.0** and **B.1** lead us to the following two theorems. In these theorems, the decorated inequality in

$$X(n) \underset{k}{<} Y(n)$$

means that $X(n) \le Y(n)$ for all $n \ge 0$ and that $X(n) < Y(n)$ for $n \ge k$. The same goes for $\underset{k}{>}$.

Theorem

For all $n \ge 0$,

$$N(1, 9, 3n + 2) \underset{2}{>} N(0, 9, 3n + 2),$$

$$N(0, 9, 3n + 1) > N(2, 9, 3n + 1),$$

$$N(2, 9, 3n) \underset{4}{>} N(3, 9, 3n).$$

Theorem

For all $n \geq 0$,

$N(0,8,4n+1) \geq_2 M(0,8,4n+1)$, $N(0,8,4n+2) \leq_2 M(0,8,4n+2)$.

$N(0,8,4n+3) \geq_1 M(0,8,4n+3)$, $N(0,8,4n+1) \geq_6 M(1,8,4n+1)$,

$N(0,8,4n+3) \geq_{10} M(1,8,4n+3)$, $N(0,8,4n+1) > M(2,8,4n+1)$,

$N(0,8,4n+2) < M(2,8,4n+2)$, $N(0,8,4n+3) > M(2,8,4n+3)$,

$N(0,8,4n+1) > M(4,8,4n+1)$, $N(0,8,4n+2) \leq_1 M(4,8,4n+2)$,

$N(0,8,4n+3) > M(4,8,4n+3)$, $N(1,8,4n+2) > M(1,8,4n+2)$,

$N(1,8,4n+2) \geq_2 M(4,8,4n+2)$, $N(2,8,4n) < M(0,8,4n)$,

$N(2,8,4n) \leq_1 M(2,8,4n)$, $N(2,8,4n) \leq_1 M(4,8,4n)$,

$N(3,8,4n) \leq_4 M(0,8,4n)$, $N(3,8,4n) \geq_1 M(1,8,4n)$,

$N(3,8,4n+1) < M(1,8,4n+1)$, $N(3,8,4n) \geq_1 M(3,8,4n)$,

$N(3,8,4n+1) \leq_1 M(3,8,4n+1)$, $N(3,8,4n+1) \geq_1 M(4,8,4n+1)$,

$N(3,8,4n+1) < M(1,8,4n+1)$, $N(3,8,4n) \geq_1 M(3,8,4n)$,

$N(3,8,4n+1) \leq_1 M(3,8,4n+1)$, $N(3,8,4n+1) \geq_1 M(4,8,4n+1)$,

$N(4,8,4n+3) \geq_1 M(2,8,4n+3)$, $N(4,8,4n+3) \leq_{10} M(3,8,4n+3)$,

$N(4,8,4n+3) \geq_2 M(4,8,4n+3)$.