4. Odds and ends.

4.0 When Dyson asked "why **0.1.0** - **0.1.2**?", he was not, I suggest, entirely satisfied with the answer he gave and with the proof, supplied by Atkin and Swinnerton-Dyer, that his answer was correct. The methods of **[A+SD]** seem decidedly elaborate compared with the simplicity of the problem they attack.

Let us denote by $\mathbf{N}(r, m, n)$ the set of partitions of n whose ranks are congruent to r modulo m. Then a really satisfactory answer to Dyson's question would be explicit (and hopefully clear and simple) constructions of bijections

$$\mathbf{N}(r, 5, 5n+4) \simeq \mathbf{N}(r', 5, 5n+4)$$

and

$$\mathbf{N}(r, 7, 7n+5) \simeq \mathbf{N}(r', 7, 7n+5).$$

So far as I know, no such bijections have yet been found. (I believe George Andrews offers $5, or some such sum, for their discovery.)

A related question is that of the existence of bijections

$$\mathbf{M}(r, 5, 5n+4) \simeq \mathbf{M}(r', 5, 5n+4),$$

$$\mathbf{M}(r, 7, 7n+5) \simeq \mathbf{M}(r', 7, 7n+5)$$

and

$$\mathbf{M}(r, 11, 11n+6) \simeq \mathbf{M}(r', 11, 11n+6).$$

These bijections may be easier to find, since the proofs of the corresponding statements about the M's are rather more direct than those of the equalities between the N's (however, Dyson's account **[Dys1]** of **0.2.3** is less complicated than his account **[Dys2]** of **0.3.7**).

A combinatorial proof of **1.1.7** would be of considerable help in tracking down these bijections. Such a proof should not be too difficult to find (though I have not yet succeded).

However, combinatorial proofs of **0.1.0**, **0.1.1** and **0.1.2** and of the $q = 5$, $n = 2$ case of **0.1.3** have been given by Garvan, Kim and Stanton **[G+K+S]**. In their paper, they show, in a combinatorial way, how the dihedral group of order 10 may be made to act on the set of partitions of $5n+4$ in such a way that one (and ∴ all) of the 5-cycles in this group has no fixed points. thus proving **0.1.0**. Moreover, they show how to attach to each partition π of $5n+4$ a number ***crank***(π) (they call it a ***crank statistic***) such that, if σ is this 5-cycle,

$$\textit{crank}(\sigma\pi) \equiv 1 + \textit{crank}(\pi) \bmod 5.$$

And so, if $L(r, m, n)$ denotes the number of partitions of n with crank statistics congruent to r mod m,

$$L(0, 5, 5n+4) = L(1, 5, 5n+4) = \ldots = L(4, 5, 5n = 4).$$

They do just the same for the partitions of $7n + 6$, respectively $11n + 6$, using instead the dihedral groups of orders 14, respectively 22.

For t odd, define the mod t polynomial

$$p_t(x) := \left(x - (t-1)/2\right)^{t-3}.$$

Then Garvan , Kim and Stanton **[G+K+S, theorem 3]** show that, for t = 5, 7 and 11, the expressions

$$\mathrm{crank}_t(\pi) := \sum_{r \geq 0} \left(p_t(r - \pi_r) - p_t(r)\right) \qquad \textbf{4.0.0}$$

are crank statistics for partitions of tn + s (s = 4, 5 and 6, respectively).

The function crank_t defined at **4.0.0** satisfies

$$\mathrm{crank}_t(\overline{\pi}) = -\,\mathrm{crank}_t(\pi) \qquad \textbf{4.0.1}$$

(c.f. **0.3.9**). Noting that the function crank_t satisfies **4.0.2** (below), this follows from the simple (but surprising ?)

Lemma

Suppose f is any function defined on the integers, taking values in any abelian group, that enjoys the property

$$f(-1 - x) = f(x) \qquad \textbf{4.0.2}$$

for every integer x. Then, for any partition π,

$$\sum_{r \geq 0} \left(f(r - \pi_r) - f(r)\right) = -\sum_{s \geq 0} \left(f(s - \pi_s) - f(s)\right) \qquad \textbf{4.0.3}$$

Proof Represent the partition π as the set

$$\pi := \left\{(r, s) \in \mathbb{Z}^2 : 0 \leq r < \pi_s,\ 0 \leq s < |\pi|\right\}.$$
$$= \left\{(s, r) \in \mathbb{Z}^2 : 0 \leq r < \overline{\pi}_s,\ 0 \leq s < |\overline{\pi}|\right\}$$

and define

$$g(r,s) := \mathfrak{f}(r-s-1) - \mathfrak{f}(r-s).$$

Then **4.0.2** gives

$$\mathcal{g}(r,s) = -\mathcal{g}(s,r)$$

and so we have

$$\sum_{r \geq 0}\left(\mathfrak{f}(r-\pi_r) - \mathfrak{f}(r)\right) = \sum_{(r,s) \in \pi} \mathcal{g}(r,s)$$

$$= -\sum_{(r,s) \in \pi} \mathcal{g}(s,r) = -\sum_{(s,r) \in \pi} \mathcal{g}(r,s)$$

$$= -\sum_{s \geq 0}\left(\mathfrak{f}(s-\pi_s) - \mathfrak{f}(s)\right). \qquad \boxed{\text{qed}}$$

4.0.3 must be well known. It probably appears, perhaps in a different form, in **[Mac]**.

Garvan, Kim and Stanton also give a crank statistic for the partitions of $25n+24$ **[G+K+S, theorem 6]**, thus giving a combinatorial proof of the $q = 5$, $n = 2$ case of **0.1.3**. This crank statistic is too unwieldy to give here.

These combinatorial proofs do not explicitly give bijections of the type described above. No doubt such bijections could be wrung out of these proofs, but I suspect they might be rather cumbersome.

4.1 So far as I am aware, no combinatorial proof of a fact about Dyson's rank has yet appeared in print (apart, that is, from Dyson's own derivations of the generating functions for the rank and the crank). To remedy this state of affairs, I give in this section a combinatorial proof of

Theorem

$$N(0,2,2n) < N(1,2,2n), \text{ for } n \geq 1, \tag{4.1.0}$$

$$N(0,2,2n+1) > N(1,2,2n+1), \text{ for } n \geq 0. \tag{4.1.1}$$

First I shall give a simple algebraic proof of **4.1.0** and **4.1.1**. It is, I think, plain that

$$\sum_{m,n} N(m,n) z^m q^n = 1 + \sum_{k \geq 1} \frac{z^{k-1} q^k}{(z^{-1}q;q)_k}, \tag{4.1.2}$$

where

$$(z;q)_k := (z;q)/(zq^k;q).$$

Setting $z = -1$ in **4.1.2** gives

$$\sum \big(N(0,2,n) - N(1,2,n)\big) q^n = 1 + \sum_{k \geq 1} (-)^{k-1} \frac{q^k}{(-q;q)_k}. \tag{4.1.3}$$

Now

$$(-q;q)_k^{-1} = \prod_{\substack{1 \leq r \leq (k+1)/2 \\ (k+1)/2 \leq s \leq k}} (1 - q^{2r-1})(1 - q^{2s})^{-1} \tag{4.1.4}$$

$$= \sum_{n \geq 0} a_{k,n} q^n, \text{ say,}$$

and it is clear that

$$a_{k,2n} \geq 0, \quad a_{k,2n+1} \leq 0 \tag{4.1.5}$$

and that these inequalities are strict when $k = 1$ ($a_{1,n} = (-1)^n$).

Now, from **4.1.3**,

$$N(\mathcal{O},2,n) - N(1,2,n) = \sum_{k=1}^{n} (-)^{k-1} a_{k,n-k}\ , \text{ for } n \geq 1,$$

and, by **4.1.5**, this sum is strictly positive/negative according as n is odd/even.

So **4.1.0** and **4.1.1** are proved.

For $\varepsilon = 0$ or 1 and k a positive integer, set

$$\mathcal{R}(\varepsilon) := \{\text{partitions } \pi \neq \mathcal{O} : w(\pi) + \pi_0 + |\pi| \equiv \varepsilon \bmod 2\},$$

$$\mathcal{R}(k,\varepsilon) := \{\pi \in \mathcal{R}(\varepsilon) : \pi_0 = k\}.$$

I now show how to construct an injective weight-preserving map

$$\Phi : \mathcal{R}(0) \longrightarrow \mathcal{R}(1),$$

which will establish the versions of **4.1.0** and **4.1.1** with weak inequalities.

Let

$$\mathcal{E}(k,\varepsilon) := \{\text{partitions } \pi : \pi_0 \leq k,\ w(\pi) + |\pi| \equiv \varepsilon \bmod 2\}.$$

I construct the map Φ by constructing injective, weight-preserving maps

$$\Psi_k : \mathcal{E}(k,1) \longrightarrow \mathcal{E}(k,0), \qquad \textbf{4.1.6}$$

extending each Ψ_k to

$$\Phi_k : \mathcal{R}(k,0) \longrightarrow \mathcal{R}(k,1)$$

in the obvious way (i.e. by adjoining a top part k), and piecing together these Φ_k.

All I have to show then is how to construct the maps Ψ_k. This I do by using the "involution principle" of **[G+M]** to "combinatorialise" (dreadful word) **4.1.5**.

Suppose X is a ***WS-set* [Lew1]**, that is, a set each of whose elements carries a *sign*

$$\operatorname{sign} x = + \text{ or } -$$

and a *weight*

$$w(x) = \text{a non-negative integer.}$$

I also want X to be *locally-finite* (we don't in fact need quite as much as this), meaning that each subset $\{x : w(x) = n\}$ is finite. If $S \subseteq X$, denote by S^+ and S^- the subsets of S of elements of sign +, respectively -.

Suppose that X is a WS - set, X^α and X^β are (locally-finite) subsets of X and that $\alpha : X^\alpha \longrightarrow X^\alpha$ and $\beta : X^\beta \longrightarrow X^\beta$ are weight-preserving and sign-changing involutions. Set $X_\alpha := X \backslash X^\alpha$, the subset of X on which α is not defined and define X_β in the same way. If $x \in X_\alpha$, it follows from the finiteness condition that the sequence

$$x,\ \beta x,\ \alpha\beta x,\ \beta\alpha\beta x,\ \ldots$$

must eventually come to rest at some point

$$gm\,x = gm_{\alpha,\beta}x := \beta^{\varepsilon}(\alpha\beta)^n x \quad (\text{where } \varepsilon = 0 \text{ or } 1)$$
$$\in \begin{cases} X_\beta, & \text{if } \varepsilon = 0 \\ X_\alpha, & \text{if } \varepsilon = 1. \end{cases}$$

Moreover, since α and β are sign-changing,

$$\left.\begin{aligned} x \in X_\alpha^+ &\Longrightarrow gm\,x \in X_\alpha^- \cup X_\beta^+ \\ x \in X_\alpha^- &\Longrightarrow gm\,x \in X_\alpha^+ \cup X_\beta^- \end{aligned}\right\} \qquad \textbf{4.1.7}$$

and *gm* is injective. *gm* is also plainly weight-preserving.

If X and Y are WS-sets, we regard the cartesian product $X \times Y$ as a WS-set, with

$$\mathrm{sign}(x, y) = (\mathrm{sign}\, x).(\mathrm{sign}\, y) \text{ and } w(x, y) = w(x) + w(y).$$

If $\mathcal{Y}$ is a set of partitions, we also regard $\mathcal{Y}$ as a WS - set, taking the weight of a member of $\mathcal{Y}$ as the sum of its parts and setting

$$\mathrm{sign}\, \pi := (-)^{w(\pi)}.$$

We also define another WS - set, $\Sigma\mathcal{Y}$, with underlying set $\mathcal{Y}$ and the weights as before, but now

$$\mathrm{sign}\, \pi := (-)^{w(\pi) + |\pi|}.$$

For W a set of positive integers, define

$$\mathcal{E}(W) := \{\text{partitions } \pi : \text{each } \pi_i \in W\}$$
$$\mathcal{E}_d(W) := \{\pi \in \mathcal{E}(W) : \pi_0 > \pi_1 > \pi_2 > \ldots\}$$

The ***cancelling involution* [G+M]**, γ, is defined on the WS-set $\mathcal{E}(W) \times \Sigma\mathcal{E}_d(W)$ by:

$$\gamma(\emptyset, \emptyset) = (\emptyset, \emptyset)$$

and, if $(\pi, \rho) \neq (\emptyset, \emptyset)$,

$$\gamma(\pi, \rho) := (\pi', \rho')$$

where, if $\pi_0 > \rho_0$, π_0 is removed from π and thrown into ρ and otherwise ρ_0 is moved from ρ to π (I described a special case of this involution earlier, just after

0.3.4). γ is a weight-preserving involution, sign-changing away from $(\emptyset, \emptyset)$. The same construction gives a sign-changing involution on $\sum \mathcal{E}(W) \times \mathcal{E}_d(W)$ (also called a cancelling involution).

The cancelling involution represents a realisation of the identity

$$\prod_{w \in W} \frac{1}{1 - q^w} \times \prod_{w \in W} (1 - q^w) = 1$$

Let

$$\mathcal{E}(m) := \mathcal{E}(m, 0) \cup \mathcal{E}(m, 1) = \{\text{partitions } \pi : \text{each part} \le m\},$$

$$\mathcal{O}(m) := \{\text{partitions } \pi : \text{each part odd and} \le m\},$$

$$\mathcal{B}(m) := \{\text{partitions } \pi : \pi_i \text{ even, } m < \pi_i \le 2m \text{ for } 0 \le i < |\pi|\}.$$

and let $\mathcal{E}_d(m)$, $\mathcal{O}_d(m)$ and $\mathcal{B}_d(m)$ denote the sets of partitions with the same restrictions and their parts distinct. There's a weight-preserving bijection

$$\varphi : \mathcal{E}_d(m) \times \mathcal{B}(m) \longrightarrow \mathcal{O}(m)$$

defined on a pair (π, ρ) by splitting each even part (if any) of π and of ρ equally in two over and over again until we have only odd numbers. These odd numbers make up $\varphi(\pi, \rho) \in \mathcal{O}(m)$. The inverse to φ is defined on $\sigma \in \mathcal{O}(m)$ by adding equal parts of σ over and over again until (if ever) a number larger than m appears. This number is thrown into the right-hand component of $\varphi^{-1}\sigma$ and the process is repeated on the rest of σ. When this operation can no longer be performed, we have distinct numbers ($\le m$) remaining in σ and they make up the left-hand component of $\varphi^{-1}\sigma$. This construction is due essentially to Glaisher **[Gla]**. It is a combinatorial version of **4.1.4** (turned upside-down, then divided by $\Pi(1-q^{2s})\big)$. In these two constructions, it is irrelevant in which order the operations are performed.

Define the WS-set

$$\mathcal{X}(m) := \Sigma\mathcal{E}(m) \times \mathcal{O}(m) \times \Sigma\mathcal{B}_d(m) \times \Sigma\mathcal{O}_d(m) \times \mathcal{B}(m)$$

and define sign-changing involutions α and β on $\mathcal{X}(m)$ in the following way. α acts on $(\pi, \rho, \sigma, \tau, \upsilon)$ by the cancelling involution on (ρ, τ) (leaving π, σ and υ alone), if $(\rho, \tau) \neq (\mathit{0}, \mathit{0})$, and otherwise α does the cancelling involution on (σ, υ). α is <u>not</u> defined on

$$\mathcal{X}(m)_\alpha = \Sigma\mathcal{E}(m) \times \{\mathit{0}\} \times \{\mathit{0}\} \times \{\mathit{0}\} \times \{\mathit{0}\}.$$

To define the action of β, we first identify $\mathcal{O}(m)$ with $\mathcal{E}_d(m) \times \mathcal{B}(m)$ via the bijection φ, identifying ρ with (ρ', ρ''), say. Then β does the cancelling involution on (ρ'', σ), if $(\rho'', \sigma) \neq (\mathit{0}, \mathit{0})$ and, when $(\rho'', \sigma) = (\mathit{0}, \mathit{0})$, β does the cancelling involution on (π, ρ'). β is undefined on

$$\mathcal{X}(m)_\beta = \{\mathit{0}\} \times \{\mathit{0}\} \times \{\mathit{0}\} \times \Sigma\mathcal{O}_d(m) \times \mathcal{B}(m)$$

Now it is plain that

$$\mathcal{X}(m)_\beta^- = \varnothing,$$

so, identifying $\mathcal{X}(m)_\alpha^-$ with $\Sigma\mathcal{E}(m)^- = \mathcal{E}(m, 1)$, **4.1.7** shows that $\Psi_m = \mathit{gm}$ satisfies **4.1.6**. (To establish the strictness of the inequalities **4.1.0** and **4.1.1**, it is enough to show that none of the partitions $111\ldots1 \in \mathcal{R}(1)$ lies in the image of Φ. I leave the proof of this fact as an easy exercise for the reader.)

Take, for example, the partition 6442, which lies in $\mathcal{R}(0)$. To find $\Phi(6442) = \Phi_6(6442)$ we have first to find $\Psi_6(442)$. The α-β sequence defining gm and $\therefore$ Ψ_6 runs :

$$(442,0,0,0,0) \xrightarrow{\beta} (42,1111,0,0,0) \xrightarrow{\alpha} (42,111,0,1,0) \xrightarrow{\beta} (2,1111111,0,1,0)$$
$$\xrightarrow{\alpha} (2,11111111,0,0,0) \xrightarrow{\beta} (2,0,8,0,0) \xrightarrow{\alpha} (2,0,0,0,8) \xrightarrow{\beta} (0,11,0,0,8) \xrightarrow{\alpha} (0,1,0,1,8)$$
$$\xrightarrow{\beta} (1,0,0,1,8) \xrightarrow{\alpha} (1,1,0,0,8) \xrightarrow{\beta} (11,0,0,0,8) \xrightarrow{\alpha} (11,0,8,0,0) \xrightarrow{\beta} (11,11111111,0,0,0)$$
$$\xrightarrow{\alpha} (11,1111111,0,1,0) \xrightarrow{\beta} (411,111,0,1,0) \xrightarrow{\alpha} (411,1111,0,0,0) \xrightarrow{\beta} (4411,0,0,0,0)$$

at which point it comes to rest. So $\Psi_6(442) = 4411$ and $\therefore$ $\Phi_6(6442) = 64411$. (Note that the maps Ψ_k depend on k. For example, $\Psi_8(442) = 82$)

When $X_\alpha = X_\alpha^+$ and $X_\beta = X_\beta^+$, $gm = gm_{\alpha,\beta}$ is an isomorphism, with inverse $gm_{\beta,\alpha}$. This is the case discussed by Garsia and Milne **[G+M]**. In their proof of the first Rogers-Ramanujan identity, they construct a signed set X and sign-changing involutions α and β on X, for which X_α is (isomorphic to) the set of partitions of n with parts $\equiv \pm 1$ mod 5, all with sign +, and X_β is (isomorphic to) the set of partitions of n whose parts differ by 2, each again having positive sign. The map gm then gives a bijection between these two sets.

4.2 I am sure that the methods of §§**3.1-3.3** could be applied to prove theorems A, B, C, F and G and also any other theorems of this type (if, indeed, there are any others apart from trivial consequences thereof). In partial justification of this claim, I shall show in this section how one might set about trying to establish a linear relation suspected to hold amongst the numbers $N(r, p, pn+s)$ and $M(r', p, pn+s)$ in the cases when p is prime and greater than 3.

Take, then, a prime p greater than 6. I shall look first at the functions $\mathcal{P}_k := \mathcal{P}_k^{(p)}$ (defined at **3.1.8**) and I shall show that, up to multiplication by roots of unity, $\Gamma_0(2p)$ acts as a group of permutations on these $\mathcal{P}_k$ $\left(\text{by } (\bullet\,|_{-1} A)\right)$ and, in particular, that each $\mathcal{P}_k$ is a modular form of weight -1/2 on $\Gamma_1(2p)$. Throughout this section (this may not everywhere be necessary) A denotes a matrix in $\Gamma_0(2p)$.

We have

$$\begin{aligned}\mathcal{P}_k &:= (1/p)\sum \exp(r\pi i/12p)\exp(-2kr\pi i/p)\eta\big((\tau+r)/p\big)^{-1}\\ &= (1/p)\sum \exp(r\pi i/12p)\exp(-2kr\pi i/p)\mathfrak{p}_r(\tau)\end{aligned}$$

where, with

$$J_p := \begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix} \text{ and } U := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

I set

$$\mathfrak{p}_r(\tau) := \eta(J_p U^r \tau)^{-1}.$$

For $r \in \{0, 1, \ldots, p-1\}$, let $s \in \{0, 1, \ldots, p-1\}$ be that integer such that

$$s \equiv bd + rd^2 \bmod p \ (\text{so } sa \equiv b + rd \bmod p) \qquad \textbf{4.2.0}$$

(note that the association $r \mapsto s$ is, of course, bijective) and define

$$A_r := U^r A U^{-s} = \begin{pmatrix}a + rd & -rsc - sa + b + rd\\ c & d - sc\end{pmatrix}.$$

Then we have

$$\left(\mathfrak{p}_r\,|_{-1}\,A\right)(\tau) = \varepsilon(J_pA_rJ_p^{-1})^{-1}\varepsilon(A)\mathfrak{p}_s(\tau),$$

so the action of A permutes the $\mathfrak{p}_r$ (up to multiplication by roots of unity). What we have to show is that, for each $0 \le k < p$, there's an integer $0 \le k_A < p$, with $k_A = k$ when $A \in \Gamma_1(2p)$, such that

$$\begin{aligned}\Delta(p,k;A) &:= \exp\left(\pi i(r-s)/12p\right)\exp\left(-2(kr-k_As)/p\right)\varepsilon(J_pA_rJ_p^{-1})^{-1}\varepsilon(A)\\ &= \exp\left(\pi i\Omega(p,k;A)/12\right),\end{aligned} \qquad \textbf{4.2.1}$$

say, is independent of r. (Also that the association $k \mapsto k_A$ is bijective.)

For coprime c, d, with d odd, set

$$j(c,d) := 6\left(1 - \left(\tfrac{c}{d}\right)_*\right)$$

and note the simple consequence of the law of quadratic reciprocity that, when d is odd and c is even,

$$\begin{aligned}j(p,d-c) &= j(p,d), \text{ if } p \equiv 1 \bmod 4 \text{ or } c \equiv 0 \bmod 4\\ &= 12 - j(p,d) \text{ otherwise.}\end{aligned} \qquad \textbf{4.2.2}$$

Defining

$$\Psi(A) := (a+d)c + bd(1-c^2) + 3d(1-c) - 3 + j(c,d),$$

we have, by **2.4.1**,

$$\varepsilon(A) = \exp\left(\pi i\Psi(A)/12\right),$$

if d is odd. If Φ is the function on $SL_2(\mathbb{Z})$ defined by Rademacher **[Rad, 71.2]**, we have

$$\Psi(A) = \Phi(A), \text{ if } c = 0 \text{ and } \Psi(A) = \Phi(A) + 3, \text{ otherwise.}$$

It then follows from **[Rad, 71.6]** that

$$\Psi(U^m A U^{-n}) = \Psi(A) + m - n. \qquad \textbf{4.2.3}$$

Suppose A has $b \equiv 0 \bmod p$. Then it is easy to see from the definitions that

$$\Psi(J_p A J_p^{-1}) = \Psi(A) + \Big((-a+2d)c + bd/p + bdc^2\Big)(1-p) + j(p, d). \qquad \textbf{4.2.4}$$

Now, for any A in $\Gamma_0(2p)$, throwing together **4.2.3** and **4.2.4** gives

$$\Psi(J_p A_r J_p^{-1}) = \Big((\ldots)c^2 + \big((s^2-p)a + 2(pd-sb-rsd)\big)c/p + (-s+bd+rd^2)/p\Big)(1-p) + j(p, d-sc) + r - s + \Psi(A). \qquad \textbf{4.2.5}$$

and, from **4.2.5** and **4.2.1**, we have

$$\Omega(p, k; A) = \Big(-(\ldots)c^2 - \big((s^2-p)a + 2(pd-sb-rsd)\big)c/p - (bd - r + rd^2)/p\Big)(1-p) + 24(k_A s - kr)/p + j(p, d-sc). \qquad \textbf{4.2.6}$$

Suppose first that $p \equiv 1 \bmod 6$ and consider the two possibilities :

(i) $(p-1)c \equiv 0 \bmod 24$. In this case, choose $k_A \in \{0, 1, \ldots, p-1\}$ so that

$$k_A \equiv ka^2 + (p-1)(a^2-1)/24 \mod p$$

(the right-hand side here is an integer, since a is odd). From **4.2.6**, the choice of s at **4.2.0** and **4.2.2**, we have

$$\Omega(p, k; A) = (-1 + 24k + p)ab/p + j(p, d) + 24 \times \text{integer} \qquad \textbf{4.2.6}$$

and $\Delta(p, k; A) = \exp\big(\pi i \Omega(k; A)/12\big)$ is free from r.

(ii) $c(p-1) \equiv 12 \bmod 24$, so $p \equiv 3 \bmod 4$ and $c/2$ is odd. With the same choice of k_A, we find that

$$\Omega(p,k;A) = (-1+24k+p)ab/p) + (p-s^2)ac(1-p)/p + j(p,d-sc) + 24 \times \text{integer}$$

$$= (-1+24k+p)ab/p + j(p,d) + 12 + 24 \times \text{integer}, \qquad \mathbf{4.2.7}$$

by **4.2.2**, and again $\Delta(p,k;A)$ is free from r.

The argument is not much different when $p \equiv -1 \bmod 6$. By looking at the expression for $\Psi(L_p A L_p^{-1}) + \Psi(A)$ instead of that for $\Psi(L_p A L_p^{-1}) - \Psi(A)$ we find that

$$\Omega(p,k;A) = \Big(-(\ldots)c^2 - \big((s^2+p)a - 2(pd + sb + rsd)\big)c/p - $$
$$- (bd - r + rd^2)/p\Big)(1+p) + 6(1-d+sc)$$
$$+ (24kr - 24k_A s)/p + j(p,d-sc) + 2\Psi(A).$$

With much the same choice of k_A, viz.

$$k_A \equiv k - (p+1)(a^2-1)/24 \bmod p,$$

we have

$$\Omega(p,k;A) = (-1+24k-p)ab/p - (s^2+p)ac(1+p)/p + j(p,d-sc)$$
$$+ 6sc + 6(1-d) + 24 \times \text{integer}$$

and we have, with the help of **4.2.2**,

$$\Omega(p,k;A) = (-1+24k-p)ab/p + j(p,d) + 6(1-d) + 24 \times \text{integer}, \qquad \mathbf{4.2.8}$$

if $(p+1)c \equiv 0 \bmod 24$, and

$$\Omega(p,k;A) = (-1+24k-p)ab/p + j(p,d) + 6(1-d) + 12$$
$$+ 24 \times \text{integer}, \qquad \mathbf{4.2.9}$$

when $(p+1)c \equiv 12 \bmod 24$.

We summarise **4.2.6 - 4.2.9** as follows. If p is a prime greater than 3 and A lies in $\Gamma_o(2p)$,

$$\left(\mathcal{P}_k^{(p)} \mid_{-1} A\right) = o(\mathcal{P}_k^{(p)}; A)\mathcal{P}_{k_A}^{(p)}$$

where

$$k_A \equiv a^2k + \left(1 - \chi_6(p)p\right)(1-a^2)/24 \bmod p$$

(χ_6 is defined at **2.5.16**) and

$$o(\mathcal{P}_k; A) := (-)^{(1-\chi_6(p)p)c/4_1(1-\chi_6(p))(1-d)/2}\left(\frac{p}{d}\right) \times$$
$$\exp\left(2\pi i(\operatorname{ord}\mathcal{P}_k + \chi_6(p)/24)ab\right)\varepsilon(A)^{1-\chi_6(p)} \qquad \textbf{4.2.10}$$

$\left(\text{because } \operatorname{ord}\mathcal{P}_k = (k - 1/24)/p\right)$.

[Returning to this dissertation 6 months after it was presented, I now realise that the number p above need not be prime and that all we need is that p be coprime to 6. I also think that the number 2 in $\Gamma_o(2p)$ and $\Gamma_1(2p)$ is superfluous.

Indeed I believe rather more is true. Suppose $\mathcal{F}$ is a modular form on $SL_2(\mathbb{Z})$ of weight k/2. If m is a positive integer, define the functions $\mathcal{F}_r^{(m)}$ (for $0 \le r < m$) by

$$\mathcal{F}_r^{(m)}(\tau) := (1/m)\sum_{s=0}^{m-1}\exp(sk\pi i/12m)\exp(-2sr\pi i/m)\mathcal{F}\left((\tau+s)/m\right)$$

Theorem ???

Set $m' = m, 4m, 3m, 12m$ according as $(m, 6) = 1, 2, 3$ or 6. Then, up to multiplication by roots of 1, the group $\Gamma_0(m')/\Gamma_1(m')$ acts on the set $\{\mathcal{F}_r^{(m)} : 0 \le r < m\}$ by $(\bullet|_k A)$. So each $\mathcal{F}_r^{(m)}$ is a modular form on $\Gamma_1(m')$. Moreover, if $(m, 6) = 1$ and t satisfies $24t \equiv 1 \bmod m$ $(0 \le t < m)$, then $\mathcal{F}_t^{(m)}$ is a modular form on $\Gamma_0(m)$.

I am sure that this is true and that there's a proof along the lines of the above work. Alternatively, the truth of this theorem would follow from that of

Lemma ???

Suppose m is a positive integer and that c is a multiple of m' (as defined above). Then, for d and c coprime,

$$s(d + c, mc) - s(d, mc) = 2q/m + \text{even integer},$$

where q is constant on the congruence class of d mod c. When $(m, 6) = 1$, q may be taken as $(1 - \bar{d}^2)e$, where $d\bar{d} \equiv 1 \bmod m$ and $12e \equiv 1 \bmod m$.

Here, $s(\bullet, \bullet)$ is the Dedekind sum **[Rad, §68.3, p.146]**. I have some numerical evidence in support of this lemma.]

Now define the functions

$$\chi_N(m, n) = \chi_N(m, n; q) = \frac{s(m)s(2m)\eta(p)^3}{s(n+m)s(n)s(n-m)},$$

$$\chi_M(m, n) = \chi_M(m, n; q) = \frac{s(n)s(2m)\eta(p)^3}{s(n+m)s(m)s(n-m)},$$

for $m \not\equiv \pm n \bmod p$, and

$$\mathcal{X}_N(m, m) = -\frac{1}{3^g - 1} \sum_{r=0}^{g-1} 3^r \mathcal{W}_N(3^{g-r}m),$$

$$\mathcal{X}_M(m, m) = -\frac{1}{3^g - 1} \sum_{r=0}^{g-1} 3^r \mathcal{W}_M(3^{g-r}m),$$

in which g is the order of 3 modulo p and

$$\mathcal{W}_N(m) := \frac{s(4m)^3\eta(p)^3}{s(2m)^3 s(6m)} + \frac{s(2m)^3\eta(p)^3}{s(m)^3 s(3m)} - 2,$$

$$\mathcal{W}_M(m) := \frac{s(4m)^3\eta(p)^3}{s(2m)^3 s(6m)} - \frac{s(2m)^3\eta(p)^3}{s(m)^3 s(3m)}.$$

Here I've written s(k) for $s_p(k)$. Then **[A+SD, 6.7]** gives

$$S_N\big((3p-1)/2 - 3m\big) = 1 - 3m/p + (q)_\infty L(m) + \sum_{n=1}^{(p-1)/2} (-)^{m+n} \mathcal{X}_N(m, n; q^p),$$

where the L(m) are certain power series in q (similar in form to the U(m), V(m) and W(m) of §§**1.5**, **1.6** and **1.7**) whose only relevance for us is that each has the form $q^a F(q^p)$. We also have

$$S_M\big((p+1)/2 - m\big) = -m/p + \sum_{n=1}^{(p-1)/2} (-)^{m+n} \mathcal{X}_M(m, n; q^p),$$

which we obtain using arguments similar to those of §§**1.5**, **1.6** and **1.7**. Noting that, since $p \equiv \pm 1 \bmod 6$, **1.3.0** and **1.3.1** show that each $S_N(r)$ has the form

$$\pm S_N\big((3p-1)/2 - 3m\big) \text{ or } S_N\big((3p-1)/2 - 3m\big) - q^{1-r}p^{-1} - 1,$$

we have

$$\sum N(r, p, n)q^n = P/p + K + N'(r)$$

$$\sum M(r, p, n)q^n = P/p + M'(r).$$

Here, K is the sum two power series in q each of the form $q^a F(q^p)$ and, possibly, a term $\pm q^{-s}$ and

$$N'(r) = P\sum \mathcal{X}_N(m, n; q^P) \text{ and } M'(r) = P\sum \mathcal{X}_M(m, n : q^P),$$

the sums being over various (and, as we shall see, irrelevant) values of m and n.

Now we could not hope to establish a linear relation between the $N(r, p, pn + s)$ and the $M(r', p, qn + s)$ if any of the $q^a F(q^P)$ are involved in a non-trivial way (at least by the methods used here). But if this is not the case, then only the components of the $N(r)$ and $M(r')$ are involved. These are, as I shall now show, modular forms on $\Gamma_1(2p)$ and the linear relation may be confirmed by checking out the first so many cases.

Set

$$N'(r) := \sum x(r, n)q^n \text{ and } M'(r) = \sum y(r, n)q^n$$

and define

$$\mathcal{N}_k(r) := q^{(24k-1)/24p}\sum x(r, pn + k)q^n,$$

$$\mathcal{M}_k(r) := q^{(24k-1)/24p}\sum x(r, pn + k)q^n$$

Then

$$\mathcal{N}_k(r) = \sum \mathcal{P}_s \mathcal{X}_N(m, n) \quad \text{and} \quad \mathcal{M}_k(r) = \sum \mathcal{P}_s \mathcal{X}_M(m, n)$$

the sums being over various triples (s, m, n) all of which satisfy

$$\operatorname{ord} \mathcal{P}_s + \operatorname{ord} \mathcal{X}_N(m, n) = (k - 1/24)/p$$

respectively

$$\operatorname{ord} \mathcal{P}_s + \operatorname{ord} \mathcal{X}_M(m, n) = (k - 1/24)/p \qquad \textbf{4.2.11}$$

Using **2.5.0** and **2.5.10**, it may be seen that

$$o(\mathcal{X}_N(m,n); A) = \exp\left(2\pi i ab.\mathrm{ord}\,\mathcal{X}_N(m,n)\right) \qquad \textbf{4.2.12}$$

and

$$o(\mathcal{X}_M(m,n); A) = \exp\left(2\pi i ab.\mathrm{ord}\,\mathcal{X}_M(m,n)\right), \qquad \textbf{4.2.13}$$

Noting that each of the theta products $\mathcal{X}_N$ and $\mathcal{X}_M$ has sign -1, it follows from **4.2.10**, **4.2.11**, **4.2.12** and **4.2.13** that, for (fixed k and) all $0 \le r, r' < p$, the summands of the $\mathcal{N}_k(r)$ and the $\mathcal{M}_k(r')$ are mutually coherent. So every linear combination of these functions is a modular form of weight 1/2 on $\Gamma_1(2p)$.

4.3 I close with a brief account of the methods of Nicolas Santa-Gadea. As I have said, his methods are rather different from mine. In particular, modular forms play no part in his work and **1.1.7** makes no appearance. (**1.1.7** is crucial in the argument I've given in §**3.0** and, in a special form, in the work of Atkin and Swinnerton-Dyer **[A+SD]**. Indeed, I believe this elegant result could be used to provide proofs, free from the theory of modular forms, of all the theorems A-H.) Also, he does not obtain the lists in appendix B, nor the inequalities of §**3.5**.

Nicolas starts with the generating functions of the N(r, m, n) and the M(r, m, n) in the same form that I have given them. By using the relations between the mock theta functions of the third order discovered by Watson **[Wat1]** and between those of the sixth order found by Andrews and Hickerson **[A+H]**, he shows that each "group" of identities (i.e. each collection of identities that I show to be related by modular transformations) may be reduced to a single mock theta identity.

For example, Nicolas shows how the three "$\underline{\underline{6}}$"'s of theorem E reduce to the single identity

$$3T(q) = \gamma(q), \qquad \textbf{4.3.0}$$

where

$$T(q) := \frac{1}{3} + (q)_\infty^{-1} \sum_n{}' (-)^n \frac{q^{n(3n+1)/2}}{1 - q^{3n}},$$

$$\gamma(q) := \sum_{n \geq 0} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n}.$$

4.3.0 is then established by setting $\theta = 2\pi/3$ in the identity

$$\sum_{n \geq 0} \frac{(-)^n(1 + q^n(2 - 2\cos\theta)q^{n(3n+1)/2}}{1 - 2q^n\cos\theta + q^{2n}} = (q)_\infty \sum_{n \geq 0} \frac{q^{n^2}}{(e^{i\theta}q;q)_n(e^{-i\theta}q;q)_n},$$

which may be found in **[Wat1]**, four lines from the top of page 64.

A comparison of the methods used by Santa-Gadea with those used here may well throw new light on the mock theta functions.