(a) Let $y = c_1 \sin x + c_2 \cos x + e^{-x}$.

$y' = c_1 \cos x + c_2 (-\sin x) - e^{-x}$.

$y'' = -c_1 \sin x - c_2 \cos x + e^{-x}$.

Then
$$\begin{aligned} y'' + y &= c_1 \sin x + c_2 \cos x + e^{-x} \\ &\quad - c_1 \sin x - c_2 \cos x + e^{-x} \\ &= 2e^{-x}, \end{aligned}$$

and y is a soln to $y'' + y = 2e^{-x}$.

(b) $y = c_1 \sin x + c_2 \cos x + e^{-x}$.

We want $y(0) = 2$ & $y'(0) = 1$.

$y(0) = c_2 + 1 = 2, \quad c_2 = 1.$

$y' = c_1 \cos x - c_2 \sin x - e^{-x}$

$y'(0) = c_1 - 1 = 1, \quad c_1 = 2.$

So a solution is

$$y = 2 \sin x + \cos x + e^{-x}.$$

~~Ques~~:

NOTE: For a 2^{nd} order IVP the initial conditions have the form $y(x_0) = y_0, \quad y'(x_0) = y_1$.

Questions

(1) Given an IVP is there a solution?

(2) Does the IVP have a unique solution?

Theorem: Consider the IVP

(*) $\dfrac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$