2.3 Linear Equations

Ex 2.3: 1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19, 31

A first order linear DE has the form

$$a(x)\frac{dy}{dx} + b(x)y = c(x).$$

We need the following

Theorem: The general solution of the DE

$$\frac{dy}{dx} = p(x)\,y$$

is

$$y = c\,e^{\int p(x)dx}$$

where C is any constant.

Proof: Suppose $y = c\,e^{\int p(x)\,dx}$

Then $\frac{dy}{dx} = c\,e^{\int p(x)dx}\,\frac{d}{dx}\left(\int p(x)\,dx\right)$

$$= c\,e^{\int p(x)dx}\,p(x)$$

$$= p(x)\left(c\,e^{\int p(x)\,dx}\right)$$

$$= p(x)\,y.$$

So y is a solution to $\frac{dy}{dx} = p(x)\,y$.

We show all solutions have this form.

Suppose $\frac{dy}{dx} = p(x)\,y$.

Let

$$z = y\,e^{-\int p(x)\,dx}.$$

Then $\frac{dz}{dx} = e^{-\int p(x)dx}\frac{dy}{dx} + y\,(-p(x))\,e^{-\int p(x)\,dx}$

$$= e^{-\int p(x)dx}\left(\frac{dy}{dx} - y\,p(x)\right) = e^{-\int p(x)dx}(0) = 0$$

So $z = c$ (constant)

Hence $y\,e^{-\int p(x)\,dx} = c$ & $y = c\,e^{\int p(x)dx}$. $\square$

<u>Example</u> Solve $\frac{dy}{dx} = x^2 y$.

<u>Soln:</u>

$$y = c\, e^{\int x^2 dx} = c\, e^{x^3/3}.$$

The general soln is given by

$$y = c\, e^{x^3/3}$$

where c is any constant.

Now we are ready to tackle the general soln of a linear DE

$$a(x) \frac{dy}{dx} + b(x) y = c(x)$$

$$\Leftrightarrow \quad \frac{dy}{dx} + \frac{b(x)}{a(x)} y = \frac{c(x)}{a(x)} \qquad (\text{assuming } a(x) \neq 0)$$

$$\Leftrightarrow \quad \frac{dy}{dx} + p(x) y = q(x) \qquad [\text{STANDARD FORM}]$$

where $p(x) = \frac{b(x)}{a(x)}$, $q(x) = \frac{c(x)}{a(x)}$.

<u>Method for solving a 1st order linear DE:</u>

(1) Get the DE in standard form:

$$(*) \quad \frac{dy}{dx} + p(x) y = q(x).$$

(2) Multiply both sides of $(*)$ by an integrating factor $\mu(x)$

$$\mu(x) \frac{dy}{dx} + \mu(x) p(x) y = q(x) \mu(x)$$

So that the LHS is

$$\frac{d}{dx} \mu(x) y = \mu(x) \frac{dy}{dx} + \mu'(x) y$$

ie We want $\mu'(x) = p(x)\mu(x)$

ie take $\mu(x) = e^{\int p(x) dx}$.

(3) The DE can be written as

$$\frac{d}{dx}\,\mu(x)\,y = q(x)\,\mu(x), \quad \text{where } \mu(x) = e^{\int p(x)dx}$$

Integrate both sides to ~~and~~ obtain

$$\mu(x)\,y = \int q(x)\,\mu(x)\,dx + C$$

(4) The general soln is given by

$$\boxed{y = \frac{1}{\mu(x)}\left\{ \int q(x)\mu(x)\,dx + C \right\},}$$

where C is any constant.

<u>Example</u>: Solve

$$\frac{dy}{dx} - \frac{y}{x} = 2x+1 \qquad (\text{for } x>0).$$

Here $p(x) = -1/x$, $q(x) = 2x+1$.

<u>STEP 1</u>: Let $\mu(x) = e^{\int p(x)dx} = e^{\int -1/x\,dx}$
$= e^{-\ln x} = \frac{1}{x}$.

<u>STEP 2</u>: Let $z = \mu(x)\,y = \frac{1}{x}y$.

Then

$$\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dx} + y\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x}\left(\frac{dy}{dx} - \frac{1}{x}y\right)$$

$$= \frac{1}{x}(2x+1) = 2 + \frac{1}{x}.$$

$$\frac{dz}{dx} = 2 + \frac{1}{x}.$$

STEP 3 $\quad z = \int 2 + \frac{1}{x}\, dx \quad + C$
$= 2x + \ln x + C$

STEP 4 $\quad z = y/x$ and so $y = xz$ and the general soln of the DE is given by

$$\boxed{y = x\,(2x + \ln x + C)} \quad \text{for } x > 0$$

where C is any constant.

Example (p. 52) A rock contains two radioactive isotopes RA_1, RA_2; RA_1 decays into RA_2 which then decays into stable atoms. Assume rate at which RA_1 decays into RA_2 is $50\, e^{-10t}$ kg/sec. Let $y(t)$ be the mass of RA_2 present at time t. The rate of decay of RA_2 is prop. to $y(t)$.

$$\frac{dy}{dt} = \text{rate of creation} - \text{rate of decay}$$

$$\frac{dy}{dt} = 50e^{-10t} - ky \qquad (k>0).$$

Assume $k=2$ and $y(0) = 40$ kg. Find $y(t)$.

$$\frac{dy}{dt} + 2y = 50e^{-10t}$$

Let $\mu = e^{\int 2dt} = e^{2t}$.
Let $z = \mu y = e^{2t} y$.
Then $\frac{dz}{dt} = e^{2t}\frac{dy}{dt} + 2e^{2t}y$

$= e^{2t}\left(\frac{dy}{dt} + 2y\right)$
$= e^{2t}\; 50e^{-10t}$
$= 50\, e^{-8t}$

$$z = \int 50 e^{-8t}\, dt$$

$$z = \frac{50 e^{-8t}}{-8} + c$$

$$e^{2t} y = -\frac{25 e^{-8t}}{4} + c$$

$y(0) = 40 \Rightarrow$

$$40 = -\frac{25}{4} + c$$

$$c = 40 + \frac{25}{4} = \frac{160+25}{4} = \frac{185}{4}.$$

$$y = e^{-2t}\left(\frac{185}{4} - \frac{25}{4} e^{-8t}\right)$$

$$y = \frac{185}{4} e^{-2t} - \frac{25}{4} e^{-10t}, \quad \text{for } t \geq 0.$$

<u>Example</u> (#22)

$$\sin x \frac{dy}{dx} + y \cos x = x \sin x \qquad y(\pi/2) = 2$$

for $0 < x < \pi$,

$$\frac{dy}{dx} + \cot x\, y = x.$$

Let $\mu(x) = e^{\int \frac{\cos x}{\sin x} dx}$

$$= e^{\ln(\sin x)} = \sin x.$$

Let $z = \mu(x) y = \sin x\, y.$

$$\frac{dz}{dx} = \sin x \frac{dy}{dx} + y \cos x = x \sin x.$$

So $z = \int \underbrace{x}_{u} \underbrace{\sin x\, dx}_{dv}$ $\quad \left(v = \int \sin x\, dx = -\cos x\right)$

$$= uv - \int v\, du = -x \cos x - \int -\cos x\, dx$$

$$= -x \cos x + \sin x + C.$$

$$\sin x \; y = -x\cos x + \sin x + C$$

$y(\pi/2) = 2 \Rightarrow$

$$(\sin \pi/2)\cdot 2 = -\pi/2 \cos \pi/2 + \sin \pi/2 + C$$

$$2 = 0 + 1 + C$$

$$C = 1.$$

So

$$y = \frac{1 + \sin x - x\cos x}{\sin x}, \quad \text{for } 0 < x < \pi.$$

<u>Existence & Uniqueness Theorem for 1^{st} order linear DEs.</u>

Let $P(x)$, $Q(x)$ be continuous functions on the open interval (a,b) which contains x_0.

Then the IVP

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0$$

has a unique solution on the interval (a,b).