2.3 Linear Equations

Ex 2.3: 1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19, 31

A first order linear DE has the form

$$a(x)\frac{dy}{dx} + b(x)\,y = c(x).$$

We need the following

Theorem: The general solution of the DE

$$\frac{dy}{dx} = p(x)\,y$$

is

$$y = C\,e^{\int p(x)dx}$$

where C is any constant.

Proof: Suppose $y = C\,e^{\int p(x)\,dx}$

Then

$$\frac{dy}{dx} = C\,e^{\int p(x)dx}\,\frac{d}{dx}\left(\int p(x)\,dx\right)$$

$$= C\,e^{\int p(x)dx}\,p(x)$$

$$= p(x)\left(C\,e^{\int p(x)\,dx}\right)$$

$$= p(x)\;y.$$

So y is a solution to $\frac{dy}{dx} = p(x)\,y$.

We show all solutions have this form.

Suppose $\frac{dy}{dx} = p(x)\;y$.

Let

$$z = y\,e^{-\int p(x)\,dx}.$$

Then

$$\frac{dz}{dx} = e^{-\int p(x)dx}\,\frac{dy}{dx} + y\,(-p(x))\,e^{-\int p(x)\,dx}$$

$$= e^{-\int p(x)dx}\left(\frac{dy}{dx} - y\,p(x)\right) = e^{-\int p(x)dx}(0) = 0$$

So $z = c$ (constant)

Hence $y\,e^{-\int p(x)\,dx} = c$ & $y = C\,e^{\int p(x)dx}$. $\square$