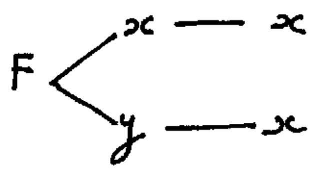
2.4 Exact Equations

Ex 2.4 odds 1–25, 14, 18

Chain rule from Calc 3

Suppose $f(x) = F(x,y)$ where $y = g(x)$.



$$f'(x) = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}$$

so $$f'(x) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$

Definition: A first order DE is <u>exact</u> if it can be written as

$$(*) \qquad \frac{d}{dx}\left(F(x,y) \right) = 0$$

where $y = y(x)$.

NOTE: (1) (*) is equivalent to the DE

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

(**) or $$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where $M = \frac{\partial F}{\partial x}$ & $N = \frac{\partial F}{\partial y}$.

(2) The general soln to (*) is given implicitly by

$$F(x,y) = c,$$

c any constant.

(3) Sometimes (**) is written as

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

<u>Example</u>: Show that the DE

$$(3x^2+y) + (5y^4+x)\frac{dy}{dx} = 0$$

is exact by showing that $\frac{\partial F}{\partial x} = 3x^2+y$

$\frac{\partial F}{\partial y} = 5y^4+x$ where $F = x^3+xy+y^5$.

Find the general soln.

<u>Soln</u>: Let $F = x^3+xy+y^5$.

$$\frac{\partial F}{\partial x} = 3x^2+y, \quad \frac{\partial F}{\partial y} = x+5y^4.$$

Hence the DE $(3x^2+y) + (5y^4+x)\frac{dy}{dx} = 0$

is exact. The DE is equivalent to

(*) $$\frac{d}{dx}\left(x^3+xy+y^5\right) = 0.$$

Hence the general soln is given implicitly by

$$x^3+xy+y^5 = c,$$

where c is any constant.

<u>CHECK</u> Assume $y=y(x)$. Then

$$\frac{d}{dx}\left(x^3+xy+y^5\right) = 3x^2+\left(x\frac{dy}{dx}+y\right)+5y^4\frac{dy}{dx}$$

$$= \left(3x^2+y\right) + \left(5y^4+x\right)\frac{dy}{dx}.$$

Theorem: If $\frac{\partial^2}{\partial x \partial y} F$ and $\frac{\partial^2}{\partial y \partial x} F$ are continuous functions then

$$\frac{\partial^2}{\partial x \partial y} F = \frac{\partial^2}{\partial y \partial x} F.$$

Suppose $M(x,y) + N(x,y) \frac{dy}{dx} = 0$ is an exact de. Then

$$M = \frac{\partial F}{\partial x} \quad \& \quad N = \frac{\partial F}{\partial y}$$

for some function F. Then

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(assuming these p.d's are continuous)

$$\boxed{\begin{array}{l} 1^{st} \text{ order p.ds of } M, N \text{ continuous} \\ \text{and } \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0 \\ \text{exact} \end{array}}$$

$$\Downarrow$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

<u>Example</u>: Consider DE

$$(x^{10/3} - 2y)\,dx + x\,dy = 0$$

$$\Leftrightarrow \quad \underbrace{(x^{10/3} - 2y)}_{M} + \underbrace{x}_{N}\frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = -2, \qquad \frac{\partial N}{\partial x} = 1 \qquad \text{The p.d are cts}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ so the DE is <u>not</u> exact.

Is converse to the Thm true?

<u>Theorem</u> If the first order partial derivatives of $M(x,y)$ & $N(x,y)$ are continuous on a rectangle R and $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then the DE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact on R.

<u>Example</u>: (#10) Determine where the DE

$$(2x+y)\,dx + (x-2y)\,dy = 0$$

is exact and solve it.

$$M = (2x+y) \qquad N = (x-2y)$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ everywhere, the p.d of M, N are continuous.
Therefore, the DE is <u>exact</u>.
There is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M = (2x+y) \text{ and } \frac{\partial F}{\partial y} = N = (x-2y)$$

$$\frac{\partial F}{\partial x} = 2x+y$$

$$F = \int (2x+y)\,dx \qquad \text{(hold } y \text{ constant \& integ. wrt } x\text{)}$$

$$= x^2 + yx + k(y).$$

$$\frac{\partial F}{\partial y} = x + k'(y) = N = x - 2y$$

So

$$k'(y) = -2y \qquad \text{(function of } y \text{ only)}$$

~~Then~~ We take $k(y) = -y^2$

Hence,

$$F = x^2 + yx - y^2 .$$

So the DE can be written as

$$\frac{d}{dx}\left(x^2+yx-y^2\right) = 0 \qquad (\text{assuming } y=y(x))$$

<u>Check</u>: Suppose $y(x)=y$. Then

$$\frac{d}{dx}\left(x^2+yx-y^2\right) = 2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx}$$

$$= (2x+y) + (x-2y)\frac{dy}{dx} .$$

Finally, the general solution is given implicitly by

$$x^2+yx-y^2 = c,$$

where c is any constant.

<u>Aside</u>: We solve this eqn for y.

$$x^2+yx-y^2 = c \qquad\qquad y^2-yx = x^2-c$$

$$\cancel{x^2-(y^2-xy) = c} \qquad\qquad y^2-yx+\frac{x^2}{4} = \frac{5x^2}{4}-c$$

$$\cancel{x^2-\left(y^2-xy+\frac{x^2}{4}\right) = c} \qquad\qquad \left(y-\frac{x}{2}\right)^2 = \left(\frac{5x^2}{4}-c\right)$$

$$y-\frac{x}{2} = \pm\sqrt{\frac{5x^2}{4}-c}, \qquad y = \frac{x}{2} \pm \sqrt{\frac{5x^2}{4}-c}.$$

<u>Method for solving an exact equation</u>

Assume $M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad (*)$

is exact; ie $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ on a rectangle R.

(1) Solve either $\frac{\partial F}{\partial x} = M$ or $\frac{\partial F}{\partial y} = N$ by integrating.

(2) Use other eqn to find F explicitly

(3) The general soln of the DE is given implicitly by $F(x,y) = c$.

<u>Example</u> (#22)

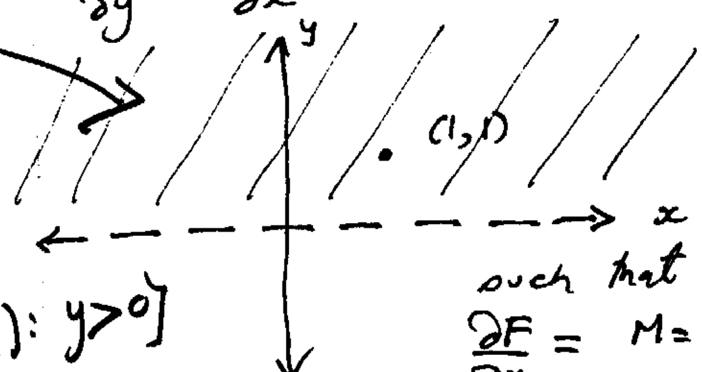
Solve the IVP

$$(ye^{xy} - 1/y)\,dx + (xe^{xy} + x/y^2)\,dy = 0,$$
$$y(1) = 1.$$

Let $M = ye^{xy} - 1/y$, $N = xe^{xy} + x/y^2$

$$\frac{\partial M}{\partial y} = y\frac{\partial}{\partial y}e^{xy} + e^{xy} + \frac{1}{y^2}$$
$$= xy\,e^{xy} + e^{xy} + \frac{1}{y^2}$$
$$\frac{\partial N}{\partial x} = xy\,e^{xy} + e^{xy} + 1/y^2$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for (x,y) satisfying $y \neq 0$.



$R = \{(x,y) : y > 0\}$

So the DE is exact for $y > 0$, since the first order partial derivatives of M, N are also continuous on R.

Assume $y > 0$.

$\exists$ function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M = (ye^{xy} - 1/y)$$

$$\& \quad \frac{\partial F}{\partial y} = N = (xe^{xy} + x/y^2).$$

$$F = \int ye^{xy} - \frac{1}{y}\,dx$$
$$= e^{xy} - \frac{x}{y} + k(y)$$
$$\frac{\partial F}{\partial y} = xe^{xy} + \frac{x}{y^2} + k'(y) = N = xe^{xy} + \frac{x}{y^2}$$
$$k'(y) = 0$$

Take $k(y) = 0$ & $\cancel{F(x,y) = e^{xy} + \frac{x}{y^2}}$

$$F(x,y) = e^{xy} - \frac{x}{y}.$$

~~The general soln is given implicitly by~~

$$\cancel{x e^{xy} + \frac{x}{y^2} = c.}$$

~~$y(1) = 1 \Rightarrow$~~

$$\cancel{1e^1 + 1 = c}$$

$$\cancel{\& \ c = e+1.}$$

~~The solution is given implicitly by the eqn~~

$$\cancel{\boxed{x e^{xy} + \frac{x}{y^2} = e+1.}}$$

The general soln is given implicitly by

$$e^{xy} - \frac{x}{y} = c,$$

where c is any constant.

$y(1)=1$ implies that

$$e^1 - 1 = c \ \&$$

The solution to the IVP is given implicitly by the equation

$$e^{xy} - \frac{x}{y} = e - 1.$$

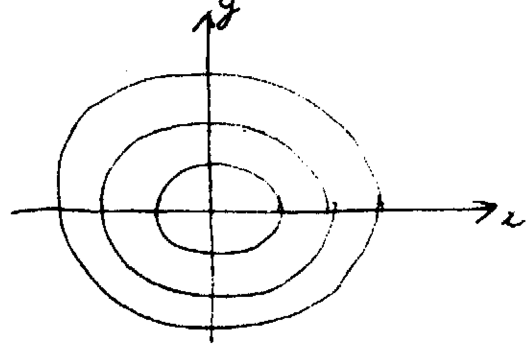
ORTHOGONAL TRAJECTORIES

Consider the family of curves

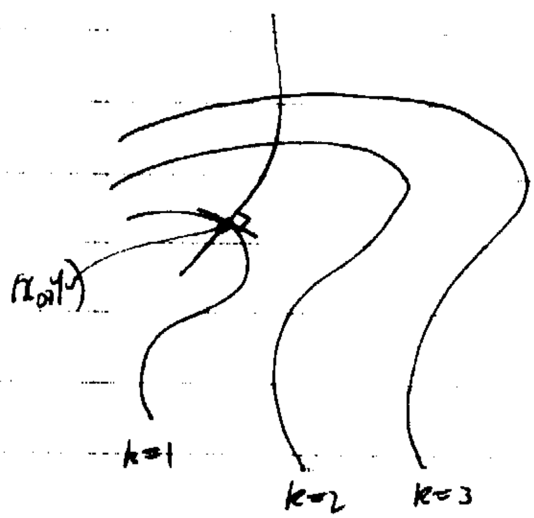
$$F(x,y) = k$$

where k is constant.

<u>Example</u> The family of curves $x^2+y^2=k$ are circles for $k>0$.



Suppose $F(x,y) = k$ where k is a constant, and $y=y(x)$.



Then $$\frac{dk}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

and $$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Suppose the graph of $y=f(x)$ is an orthogonal trajectory.

Slope of tangent to curve $y=y(x)$ at (x_0,y_0) is

$$\frac{-\frac{\partial F}{\partial x}(x_0,y_0)}{\frac{\partial F}{\partial y}(x_0,y_0)}$$

Slope of the orthogonal trajectory curve at (x_0, y_0) is

$$f'(x_0) = \frac{\frac{\partial F}{\partial y}(x_0,y_0)}{\frac{\partial F}{\partial x}(x_0,y_0)}$$

Hence the function $y=f(x)$ satisfies the DE

$$\left(\frac{\partial F}{\partial x}\right)\frac{dy}{dx} = \frac{\partial F}{\partial y}$$

or

$$\frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial x}\right)\frac{dy}{dx} = 0.$$

<u>Example</u> Find the orthogonal trajectories of

$$x^2+y^2 = k.$$

$F(x,y) = x^2+y^2$. $\quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial x} = 2x$

We must solve

$$2y - 2x\frac{dy}{dx} = 0$$

or $\quad y - x\frac{dy}{dx} = 0 \quad$ This is not exact.

But it is linear.

$$\frac{dy}{dx} - \frac{1}{x}y = 0, \quad \mu(x) = e^{-\ln x} = 1/x$$

$$\frac{d}{dx}\frac{y}{x} = \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 0$$

$$\frac{y}{x} = c \;\&\; y = cx.$$

orthogonal trajectories are straight lines thru origin.

<u>Example</u> Show that the orthogonal trajectories of the family of hyperbolas $xy = k$ are the hyperbolas $x^2 - y^2 = k$. (20)

Let $F(x,y) = xy$.

$$\frac{\partial F}{\partial x} = y \quad \& \quad \frac{\partial F}{\partial y} = x.$$

We want to ~~s~~ solve

$$x - y\frac{dy}{dx} = 0.$$

Let $M = x, \quad N = -y$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0 \quad \& \text{ this equation is exact}$$

since the partial derivatives of M, N are continuous for <u>all</u> (x,y). We want a function $G(x,y)$ such that

$$\frac{\partial G}{\partial x} = x \quad \& \quad \frac{\partial G}{\partial y} = -y.$$

$$G = x^2 + k(y)$$

$$\frac{\partial G}{\partial y} = k'(y) = -y \quad \& \text{ take } k(y) = -y^2$$

Then $\frac{d}{dx}\left(x^2 - y^2\right) = 0, \quad y = y(x)$

and orthogonal trajectories are given by

$$x^2 - y^2 = k.$$