2.5 Special Integrating Factors

Suppose a first order DE is not separable, linear or exact. What do you do?
If it is not exact is there a way to make it exact? (Sometimes).

$\mu(x,y)$ is an *integrating factor* of the DE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

if

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0$$

is exact. This implies that

$$\frac{\partial}{\partial y}\mu(x,y)M(x,y) = \frac{\partial}{\partial x}\mu(x,y)N(x,y),$$

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x},$$

and

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu.$$

<u>Example</u> (#2) Identify the DE

$$(*) \quad (2x + yx^{-1})\,dx + (xy-1)\,dy = 0$$

as separable, linear or having a integrating factor that is function of x alone or y alone.

$(*) \Leftrightarrow \quad (2x+yx^{-1}) + (xy-1)\frac{dy}{dx} = 0$. It is NOT separable & NOT linear.

(2c)

Let $M = (2x + yx^{-1})$, $N = (xy - 1)$.

$$\frac{\partial M}{\partial y} = x^{-1} \quad \& \quad \frac{\partial N}{\partial x} = y$$

so the DE is not exact (on any open region).

We try $\mu = \mu(x)$ & let

$$M = \mu(x)(2x + yx^{-1}), \quad N = \mu(x)(xy - 1).$$

Then $\frac{\partial M}{\partial y} = x^{-1}\mu(x)$ and $\frac{\partial N}{\partial x} = \mu'(x)(xy-1) + \mu(x)\, y$

~~Now try $\mu = \mu(y)$ & let~~

$$\cancel{M = \mu(y)(2x + yx^{-1}), \quad N = \mu(y)(xy - 1)}$$

$$\cancel{\frac{\partial M}{\partial y} = \mu'(y)(2x + yx^{-1}) + \mu(y)x^{-1}} \qquad \cancel{\frac{\partial N}{\partial x} = y\mu(y)}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \mu(x)(x^{-1} - y) - \mu'(x)(xy - 1)$$

$$= 0$$

if $\mu(x)(x^{-1} - y) = \mu'(x)(xy - 1)$

ie $$\frac{\mu'(x)}{\mu(x)} = \frac{x^{-1} - y}{xy - 1} = \frac{x^{-1}(1 - xy)}{(xy - 1)}$$

$$= -\frac{1}{x},$$

~~So~~ $\mu'(x) = -\frac{1}{x}\mu(x)$

and here multiplying the DE by $e^{-\ln x} = \frac{1}{x} = \mu(x)$ will make it exact, at least on any rectangular region where $x \neq 0$.

Suppose $\mu(x)$ is an integrating factor ie $\mu(x,y)$ is a function of x only. Then $\frac{\partial \mu}{\partial y} = 0$ and

$$-N \frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu$$

and $\frac{1}{\mu}\frac{\partial \mu}{\partial x} = -\frac{1}{N}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of x only. So for $\mu(x)$ to be an integrating factor we need

$$\frac{1}{N}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \text{ to be a function of } x \text{ only.}$$

Example $\quad N = (xy-1) \quad M = (2x + yx^{-1})$.

$$\frac{\partial N}{\partial x} = y \qquad \frac{\partial M}{\partial y} = \frac{1}{x}$$

$$\frac{1}{N}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{(xy-1)}\left(y - \frac{1}{x}\right)$$

$$= \frac{1}{(xy-1)} \, \frac{xy-1}{x} = \frac{1}{x}$$

is a function of x only. Since N, M have continuous partial derivatives for $x \neq 0$. The DE has an ~~integrat~~ integrating factor which is a function of x only.

Suppose $\mu(x,y) = \mu(y)$ is ~~a fn~~ an integrating factor then $\frac{\partial \mu}{\partial x} = 0$ &

$$\frac{1}{\mu}\frac{\partial \mu}{\partial y} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

is a function of y only.

Theorem If the first order partial derivatives of M, N are continuous on a rectangle &

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$ is a continuous function of y only

then $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$
has an integrating factor
which is a function of y only.

#8 Solve

$$2xy\,dx + (y^2 - 3x^2)\,dy = 0$$

$\Leftrightarrow$

$$\underbrace{2xy}_{M} + \underbrace{(y^2 - 3x^2)}_{N}\frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = -6x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -8x$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial N}{\partial y}\right) = \frac{-8x}{2xy} = \frac{-4}{y}$$ is

a function of y only.

$$\frac{d}{dy}\mu(y) = \frac{-4}{y}\mu(y)$$

$$\mu = e^{-4\ln y} = y^{-4}$$

$(*) \Leftrightarrow \quad \frac{2xy}{y^4} + \frac{(y^2 - 3x^2)}{y^4}\frac{dy}{dx} = 0 \quad (y \neq 0).$

Let $M = \frac{2x}{y^3}$, $N = \frac{1}{y^2} - \frac{3x^2}{y^4}$.

$$\frac{\partial M}{\partial y} = -\frac{6x}{y^4} \qquad \frac{\partial N}{\partial x} = -\frac{6x}{y^4}.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{for } y \neq 0.$$

The partial derivatives of M, N are continuous for $y \neq 0$. Hence the new DE is exact (on any rectangle with $y \neq 0$). We want a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = \frac{2x}{y^3} \quad \& \quad \frac{\partial F}{\partial y} = \frac{1}{y^2} - \frac{3x^2}{y^4}.$$

$$F = \int \frac{2x}{y^3}\,dx$$

$$= \frac{x^2}{y^3} + c(y)$$

$$\frac{\partial F}{\partial y} = -\frac{3x^2}{y^4} + c'(y) = \frac{1}{y^2} - \frac{3x^2}{y^4}$$

$$c'(y) = y^{-2} \quad \&$$

we take $c(y) = \frac{y^{-1}}{-1} = -\frac{1}{y}$, &

$$F(x,y) = \frac{x^2}{y^3} - \frac{1}{y}.$$

General Sol^{n} is given implicitly by

$$\frac{x^2}{y^3} - \frac{1}{y} = c$$

where c is constant.

#14 Find an integrating factor of the form $\mu = x^n y^m$ & solve

$$(12 + 5xy) + (6xy^{-1} + 3x^2)\frac{dy}{dx} = 0.$$

$$\Leftrightarrow \underbrace{(12x^n y^m + 5x^{n+1}y^{m+1})}_{M} + \underbrace{(6x^{n+1}y^{m-1} + 3x^{n+2}y^m)}_{N}\frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = 12m\, x^n y^{m-1} + 5(m+1)\, x^{n+1} y^m$$

$$\frac{\partial N}{\partial x} = 6(n+1)\, x^n y^{m-1} + 3(n+2)\, x^{n+1} y^m$$

We want $12m = 6(n+1)$ & $5(m+1) = 3(n+2)$

$$2m = n+1$$

$$5m + 5 = 3(2m+1)$$

$$5m + 5 = 6m + 3$$

$$m = 2 \;\&\; n = 3.$$

So $\mu = x^3y^2$ is an integrating factor

(*) $\Leftrightarrow$ $(12x^3y^2 + 5x^4y^3) + (6x^4y + 3x^5y^2)\frac{dy}{dx} = 0$

(provided x & $y \neq 0$).

We want $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = 12x^3y^2 + 5x^4y^3 \;\&\; \frac{\partial F}{\partial y} = 6x^4y + 3x^5y^2$$

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$$F = \int (12x^3y^2 + 5x^4y^3)\,dx$$

$$= 3x^4y^2 + x^5y^3 + c(y)$$

$$\frac{\partial F}{\partial y} = 6x^4y + 3x^5y^2 + c'(y) = 6x^4y + 3x^5y^2$$

Take $c(y) = 0$ & $F(x,y) = 3x^4y^2 + x^5y^3$.

The general soln is given ~~by~~ implicitly by

$$3x^4y^2 + x^5y^3 = c$$

where c is constant.