2.6 Substitutions & Transformations

HOMOGENEOUS EQUATIONS

A function $f(x,y)$ is called <u>homogeneous</u> if it can be written as a function of y/x alone.

<u>Example</u> Let $f(x,y) = \dfrac{y^3 + x^3 - 3xy^2}{x^2y + 7y^2x}$

$$= \frac{\frac{1}{x^3}(y^3 + x^3 - 3xy^2)}{\frac{1}{x^3}(x^2y + 7y^2x)}$$

$$= \frac{(y/x)^3 + 1 - 3(y/x)^2}{(y/x) + 7(y/x)^2}$$ which is

a function of (y/x) alone.

<u>NOTE:</u> $f(x,y)$ is homogeneous if $f(tx, ty) = f(x,y)$ for all t.

Observe if $f(tx, ty) = f(x,y)$ then
$f(x,y) = f(1, y/x)$ which is a function of y/x alone.

<u>TO SOLVE A HOMOGENEOUS DE</u>
<u>USE THE SUBSTITUTION</u> $v = y/x$
<u>WHICH MAKES THE SEPERABLE</u>

Example (#12)

Solve $x^2+y^2\,dx + 2xy\,dy = 0$

$$\Longleftrightarrow \quad x^2+y^2 + 2xy\frac{dy}{dx} = 0$$

$$2xy\frac{dy}{dx} = -x^2-y^2$$

$$\frac{dy}{dx} = \frac{-x^2-y^2}{2xy} = -\frac{1}{2}\frac{x}{y} - \frac{1}{2}\frac{y}{x}$$

Let $v = \frac{y}{x}$, $y = xv$.

$$\frac{dy}{dx} = x\frac{dv}{dx} + v = -\frac{1}{2v} - \frac{1}{2}v$$

$$x\frac{dv}{dx} = -\frac{1}{2v} - \frac{3}{2}v = -\left(\frac{1+3v^2}{2v}\right)$$

$$\int \frac{2v}{1+3v^2}\,dv = \int -\frac{1}{x}\,dx$$

$$\frac{1}{3}\ln(1+3v^2) = -\ln|x| + c$$

$$\ln(1+3v^2) = -3\ln|x| + 3c$$

$$1+3v^2 = e^{-3\ln|x|}e^{3c}$$

$$3v^2 = c'x^{-3} - 1$$

$$v^2 = cx^{-3} - 1/3$$

$$v = \pm\sqrt{cx^{-3} - 1/3}, \quad c \text{ constant}$$

$$y = \pm x\sqrt{cx^{-3} - 1/3}, \quad c \text{ constant}$$

<u>Bernoulli Equation</u> has the form

$$(\star) \quad \frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $P(x)$, $Q(x)$ are continuous functions on an interval (a,b).

Use substitution

$$v = y^{1-n}$$

$$\frac{dv}{dx} = (1-n)\,y^{-n}\,\frac{dy}{dx}$$

$$(\star) \Leftrightarrow \quad y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$$

$$\Leftrightarrow \quad \frac{1}{(1-n)}\frac{dv}{dx} + P(x)\,v = Q(x)$$

which is a linear DE.

<u>#22</u> $(*) \quad \frac{dy}{dx} - y = e^{2x}y^3$

which is a Bernoulli Eq with $n = 3$.

Let $v = y^{1-n} = y^{-2}$, so

$$\frac{dv}{dx} = -2y^{-3}\,\frac{dy}{dx}$$

$$(*) \Leftrightarrow \quad y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}$$

$$\Leftrightarrow \quad -\frac{1}{2}\frac{dv}{dx} - v = e^{2x}$$

$$dv + 2v = -2e^{2x}$$

Example Solve

$$(*) \qquad \frac{dy}{dx} - y = e^{2x} y^3.$$

This is a Bernoulli Equation with $n = 3$.

Let $\quad v = y^{1-n} = y^{-2}$ (assuming $y \neq 0$).

Then $\quad \dfrac{dv}{dx} = -2\, y^{-3} \dfrac{dy}{dx}$,

$$y^{-3} \frac{dy}{dx} = -\frac{1}{2} \frac{dv}{dx}.$$

We multiply both sides of $(*)$ by y^{-3}:

$$y^{-3} \frac{dy}{dx} - y^{-2} = e^{2x}$$

$$\Leftrightarrow \quad -\frac{1}{2} \frac{dv}{dx} - v = e^{2x},$$

which is a 1^{st} order linear DE (in $v(x)$).

$$\Leftrightarrow \quad \frac{dv}{dx} + 2v = -2e^{+2x}$$

$$\mu(x) = e^{\int 2\,dx} = e^{2x}.$$

$$\frac{d}{dx}\left(e^{2x} v\right) = e^{2x} \frac{dv}{dx} + 2e^{2x} v = e^{2x}\left(\frac{dv}{dx} + 2v\right)$$

$$\frac{d}{dx}\left(e^{2x} v\right) = -2e^{4x},$$

$$e^{2x} v = \int -2e^{4x}\, dx$$

$$e^{2x} v = -\frac{1}{2} e^{4x} + c$$

$$v = ce^{-2x} - \frac{1}{2} e^{2x},$$

$$y^{-2} = c e^{-2x} - \frac{1}{2} e^{2x}$$

<u>General Soln:</u>

$$y = \pm \left(c e^{2x} - \frac{1}{2} e^{2x}\right)^{-1/2}$$

where c is any constant (assuming $y(x) \neq 0$).