Example Find two solutions of

$$y'' - y' - 2y = 0.$$

Auxiliary Equation: $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0$$

$$r = 2, -1.$$

Two solns are $y_1 = e^{2t}$, $y_2 = e^{-t}$.

Theorem: If y_1, y_2 are solutions of

(*) $ay'' + by' + cy = 0$

then $y = c_1 y_1 + c_2 y_2$ is also a solution if c_1, c_2 are any constants.

Proof: Suppose y_1, y_2 are solns of (*). Then

$$ay_1'' + by_1' + cy_1 = 0,$$
$$ay_2'' + by_2' + cy_2 = 0.$$

Let c_1, c_2 be constants &

$$y = c_1 y_1 + c_2 y_2.$$

Then
$$y' = c_1 y_1' + c_2 y_2'$$
$$y'' = c_1 y_1'' + c_2 y_2''.$$

So

$$\begin{aligned} ay'' + by' + cy &= a(c_1 y_1'' + c_2 y_2'') \\ &\quad + b(c_1 y_1' + c_2 y_2') \\ &\quad + c(c_1 y_1 + c_2 y_2) \\ &= c_1(ay_1'' + by_1' + cy_1) \\ &\quad + c_2(ay_2'' + by_2' + cy_2) \\ &= c_1(0) + c_2(0) = 0. \end{aligned}$$

Hence $y = c_1 y_1 + c_2 y_2$ is also a soln $\square$