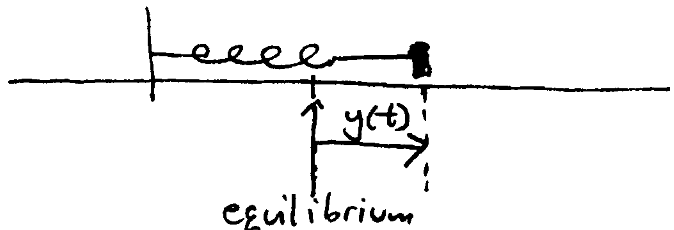
Chapter 4 Linear Second-Order Equations

4.1 Intro: mass-spring oscillator



Then $my'' + by' + ky = F_{ext}(t)$

4.2 Homogeneous Linear Equations: The general solution

Suggested homework: odds 1–19, odds 27–33, 34, 35

A linear 2^{nd} order DE has the form
$$a(t)y'' + b(t)y' + c(t)y = f(t).$$
In this chapter we will consider the case when $a(t)$, $b(t)$, $c(t)$ are constant functions, ie DE of the form
$$ay'' + by' + cy = f(t)$$
where a, b, c are constants & $a \neq 0$.

A 2nd order linear DE (with constant coefficients) is homogeneous if it has the form
$$(*) \quad ay'' + by' + cy = 0.$$

We try $y = e^{rt}$. Then $y' = re^{rt}$, $y'' = r^2e^{rt}$,
$$ay'' + by' + cy = ar^2e^{rt} + bre^{rt} + ce^{rt}$$
$$= e^{rt}(ar^2 + br + c).$$
So $y = e^{rt}$ is a solⁿ of $(*)$ iff
$$\boxed{ar^2 + br + c = 0}$$
(Auxiliary or Characteristic eqⁿ).

__Example__ Find two solutions of

$$y'' - y' - 2y = 0.$$

Auxiliary Equation: $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0$$

$$r = 2, -1.$$

Two solns are $y_1 = e^{2t}$, $y_2 = e^{-t}$.

__Theorem__: If y_1, y_2 are solutions of

(*) $ay'' + by' + cy = 0$

then $y = c_1 y_1 + c_2 y_2$ is also a solution if c_1, c_2 are any constants.

__Proof__: Suppose y_1, y_2 are solns of (*). Then

$$ay_1'' + by_1' + cy_1 = 0,$$

$$ay_2'' + by_2' + cy_2 = 0.$$

Let c_1, c_2 be constants &

$$y = c_1 y_1 + c_2 y_2.$$

Then

$$y' = c_1 y_1' + c_2 y_2'$$

$$y'' = c_1 y_1'' + c_2 y_2''.$$

So

$$\begin{aligned} ay'' + by' + cy &= a(c_1 y_1'' + c_2 y_2'') \\ &\quad + b(c_1 y_1' + c_2 y_2') \\ &\quad + c(c_1 y_1 + c_2 y_2) \\ &= c_1(ay_1'' + by_1' + cy_1) \\ &\quad + c_2(ay_2'' + by_2' + cy_2) \\ &= c_1(0) + c_2(0) = 0. \end{aligned}$$

Hence $y = c_1 y_1 + c_2 y_2$ is also a soln $\square$

Example (#16)

Find a soln to IVP

$$y'' - 4y' + 3y = 0, \qquad y(0) = 1,\ y'(0) = 1/3.$$

A.E: $r^2 - 4r + 3 = 0$

$(r-1)(r-3) = 0$

$r = 1, 3.$

Two solns are

$$y_1 = e^t, \quad y_2 = e^{3t}.$$

Let $y = c_1 e^t + c_2 e^{3t}$, $\quad c_1, c_2 \in \mathbb{R}$.

Then y is a soln of the DE.

$$y' = c_1 e^t + 3c_2 e^{3t}.$$

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = c_1 + 3c_2 = 1/3$$

$$2c_2 = -2/3, \quad c_2 = -1/3$$

$$c_1 = 1 - c_2 = 4/3.$$

So

$y = 4/3\, e^t - 1/3\, e^{3t}$ is a soln to the IVP.

NOTE: It can be shown that this is the only soln.

__Theorem__ Let $t_0, a, b, c, y_0, y_1 \in \mathbb{R}$, $a \neq 0$.
The IVP

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \; y'(t_0) = y_1$$

has a unique sol$\underline{n}$ which is valid for all t.

__Linear Independence of functions__
Two functions y_1, y_2 (defined on an interval I) are __linearly dependent__ if there are constants c_1, c_2 not both zero such that

(✱) $\quad c_1 y_1(t) + c_2 y_2(t) = 0$

for all $t \in I$. Otherwise, they are __linearly independent__.

__Example__ Let $y_1 = 2e^t$, $y_2 = 3e^t$, $I = (-\infty, \infty)$.

$$(3)(2e^t) + (-2)(3e^t) = 0$$

for all $t \in (-\infty, \infty)$. So y_1, y_2 are linearly dependent.

__Lemma__: Two functions y_1, y_2 are linearly dependent (on I) iff one of them is a constant multiple (including 0) of the other.

__Proof__ ($\Rightarrow$) Suppose y_1, y_2 are linearly dependent on I.
So there are constants c_1, c_2 not both zero such that

$$c_1 y_1(t) + c_2 y_2(t) = 0 \qquad \text{for all } t \in I.$$

Suppose $c_1 \neq 0$. Then $c_1 y_1 = -c_2 y_2$

$$y_1 = \left(\frac{-c_2}{c_1}\right) y_2$$

and y_1 is a constant multiple of y_2.

Example Determine whether $y_1 = e^{3t}$, $y_2 = e^{-4t}$ are linearly dependent on $(0,1)$.

Could $y_1 = c y_2$ for all $t \in (0,1)$ for some constant c?

If $y_1 = c y_2$ then $e^{3t} = c e^{-4t}$ for all $t \in (0,1)$
& $e^{7t} = c$ for all $t \in (0,1)$.

This impossible since e^{7t} is not a constant function.

Similarly, $y_2 \neq c y_1$ on $(0,1)$ for any constant c.

Hence y_1, y_2 are linearly independent on $(0,1)$.

Lemma: Let $a, b, c \in \mathbb{R}$, $a \neq 0$.

Let $y_1(t)$, $y_2(t)$ be two solns of

(*) $\quad ay'' + by' + cy = 0 \quad$ on (for $t \in I$).

Suppose

(**) $\quad y_1(\tau) y_2'(\tau) - y_1'(\tau) y_2(\tau) = 0 \quad$ for some $\tau \in I$.

Then y_1, y_2 are linearly dependent on I.

Proof

Suppose (*) & (**) hold.

~~Let~~ $\cancel{Y_0 = y_1(\tau),\ Y_1 = y_1'(\tau)}$.

~~Consider the IVP~~

$\cancel{(***)\ ay'' + by' + cy = 0, \quad y(\tau) = Y_0, \quad y'(\tau) = Y_1}$.

~~Then y_1 is a soln to IVP.~~

Case 1. $y_1(\tau) \neq 0$ and $y_2'(\tau) \neq 0$.

Then (**) implies $y_2(\tau) \neq 0$ & $y_1'(\tau) \neq 0$.

Let $\quad y_3 = k y_1 \quad$ where $k = \dfrac{y_2(\tau)}{y_1(\tau)}$. (6)

Then y_3 is a soln of $(*)$ because y_1 is.

$$y_3(\tau) = k y_1(\tau) = \frac{y_2(\tau)}{y_1(\tau)} y_1(\tau) = y_2(\tau).$$

$$y_3'(\tau) = k y_1'(\tau) = \frac{y_2(\tau)}{y_1(\tau)} y_1'(\tau) = \frac{y_2'(\tau) y_1(\tau)}{y_1(\tau)} = y_2'(\tau).$$

Hence y_3 & y_2 are solns of the IVP

$$ay'' + by' + cy = 0 \qquad y(\tau) = y_2(\tau),\ y'(\tau) = y_2'(\tau).$$

By uniqueness,

$$y_3 = y_2,$$
$$y_2 = k y_1 \quad \&$$

y_1, y_2 are linearly dependent.

<u>Other Cases</u> See text.

<u>NOTE</u>: The quantity $y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau)$ is called the WRONSKIAN of y_1, y_2 at τ.

<u>Theorem</u> Let $a, b, c \in \mathbb{R}$, $a \neq 0$. Let $Y_0, Y_1 \in \mathbb{R}$, $t_0 \in I$. Let y_1, y_2 be two independent solns to

$$(*) \qquad ay'' + by' + cy = 0 \qquad \text{on } I.$$

(1) The general soln to $(*)$ is $y = c_1 y_1 + c_2 y_2$ where c_1, c_2 are any constants.

(2) There exist unique constants c_1, c_2 such that $y_0 = c_1 y_1 + c_2 y_2$ is soln to the IVP

$$ay'' + by' + cy = 0, \quad y(t_0) = Y_0,\ y'(t_0) = Y_1.$$

<u>Definition</u> Let y_1, y_2 be two d'ble (differentiable) functions. The wronskian of y_1, y_2 is defined by

$$W[y_1, y_2](t) = y_1(t) y_2'(t) - y_1'(t) y_2(t)$$
$$= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

<u>Example</u> Let $y_1 = e^t$, $y_2 = e^{2t}$.

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}$$

$$= 2e^{3t} - e^{3t} = e^{3t}$$

<u>Theorem</u>: Suppose y_1, y_2 are two linearly dependent d'ble functions on an open interval I. Then

$$W[y_1, y_2](t) = 0 \qquad \text{for all } t \in I.$$

<u>Proof</u>: Suppose y_1, y_2 are dependent (& d'ble on I). So one is a constant mult. of other, say $y_2 = c y_1$, some constant c.

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & cy_1 \\ y_1' & cy_1' \end{vmatrix} = c y_1 y_1' - c y_1 y_1' = 0.$$

<u>NOTE</u>: The converse of this theorem is not true. See for example #35 in Ex 4.2.

Suppose r is a root of the auxilary eqn

$$ar^2+br+c=0 \qquad (a\neq 0)$$

(*) $\Leftarrow$ $r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Then $y=e^{rt}$ is a soln to

(**) $ay''+by'+cy=0$.

Theorem

(1) If $\Delta = b^2-4ac>0$ then (*) has two distinct real roots r_1, r_2, $y_1=e^{r_1t}$, $y_2=e^{r_2t}$ are two linearly indept. solns to (**) & the general soln of (**) is

$$y = c_1e^{r_1t}+c_2e^{r_2t},$$ where

c_1, c_2 are any constants.

(2) If $\Delta = b^2-4ac=0$ then (*) has a double root

$$r=-b/2a,$$

$y_1 = e^{-\frac{b}{2a}t}$, $y_2 = t\,e^{-\frac{b}{2a}t}$ are two indept. solns of (**) & the general soln of (**) is

$$y=(c_1+c_2t)e^{-\frac{b}{2a}t},$$ where c_1, c_2 any constants.

<u>Proof of (2)</u> Suppose $\Delta = b^2 - 4ac = 0$,
$b^2 = 4ac$.
We know
$$y_1 = e^{-bt/(2a)} \text{ is a sol}^{\text{n}} \text{ of } (**).$$
Let $y_2 = t e^{-\frac{bt}{2a}}$.
Then
$$y_2' = t\left(-\frac{b}{2a}\right)e^{-\frac{bt}{2a}} + e^{-\frac{bt}{2a}}$$
$$y_2'' = t\left(\frac{b^2}{4a^2}\right)e^{-\frac{bt}{2a}} - \frac{b}{2a}e^{-\frac{b}{2a}} - \frac{b}{2a}e^{-\frac{bt}{a}}$$

$$a y_2'' + b y_2' + c y_2 = e^{-\frac{bt}{2a}}\left(t\left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right) - b + b\right)$$
$$= 0$$
since $\frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{b^2}{4a} - \frac{b^2}{2a} + \frac{b^2}{4a} = \frac{b^2}{2a} - \frac{b^2}{2a} = 0.$

Hence $y_2 = t e^{-\frac{bt}{2a}}$ is also a sol$^{\text{n}}$ of $(**)$.
y_1, y_2 are clearly indept since
$$\frac{y_2}{y_1} = t \text{ is not constant.}$$
The result follows.

<u>Example</u> Solve
$$y'' + 6y' + 9y = 0.$$

AE: $r^2 + 6r + 9 = 0$
$(r+3)^2 = 0$
$r = -3$ is a double root.
The general sol$^{\text{n}}$ is
$$y = (c_1 + c_2 t)e^{-3t},$$ where c_1, c_2 any constants.