$$= \left(1 - \frac{\beta^2 t^2}{2!} + \frac{\beta^4 t^4}{4!} - + \cdots\right)$$

$$+ i\left(\beta t - \frac{\beta^3 t^3}{3!} + \frac{\beta^5 t^5}{5!} - + \cdots\right)$$

$$= \cos\beta t + i \sin\beta t.$$

Hence $e^{rt} = e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$

$$= e^{\alpha t}\cos\beta t + i\, e^{\alpha t}\sin\beta t.$$

Example $e^{(2+\sqrt{3}i)t} = e^{2t}\cos\sqrt{3}t + i\, e^{2t}\sin\sqrt{3}t.$

~~Theorem~~

Definition Let $f(t)$, $g(t)$ be (real) d'ble functions.
Let $z(t) = f(t) + i\, g(t)$.

$$\frac{dz(t)}{dt} := f'(t) + i\, g'(t).$$

Example
Let $z = e^{(2+\sqrt{3}i)t}$.
Find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{d}{dt}\left(e^{2t}\cos\sqrt{3}t\right) + i\frac{d}{dt}\left(e^{2t}\sin\sqrt{3}t\right)$$

$$= e^{2t}(-\sqrt{3}\sin\sqrt{3}t) + 2e^{2t}\cos\sqrt{3}t$$

$$+ i\left(e^{2t}(\sqrt{3}\cos\sqrt{3}t) + 2e^{2t}\sin\sqrt{3}t\right)$$