4.3 Auxiliary Equations with Complex Roots

Suggested HW: odds 1–25

Review:

A.E: $ar^2 + br + c = 0.$

Let $\Delta = b^2 - 4ac.$

If $\Delta < 0$ then the A.E. has two complex roots

$$r = \alpha \pm i\beta$$

where $\alpha = \frac{-b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$, $i = \sqrt{-1}$, $i^2 = -1.$

Example Find the roots of

$$r^2 - 4r + 7 = 0.$$

$$r = \frac{4 \pm \sqrt{16 - 28}}{2}$$

$$= 2 \pm \sqrt{\left(\frac{-12}{4}\right)} = 2 \pm \sqrt{-3}$$

$$= 2 \pm \sqrt{3}\, i.$$

Roots are $r = 2 + \sqrt{3}i$ & $r = 2 - i\sqrt{3}.$

Definition of e^{rt} when $r = \alpha + i\beta$

$$e^{rt} = e^{(\alpha + i\beta)t} = e^{\alpha t + i\beta t}$$

$$= e^{\alpha t} e^{i\beta t}$$

When x is real

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{i\beta t} := 1 + \frac{i\beta t}{1} + \frac{(i\beta t)^2}{2!} + \frac{(i\beta t)^3}{3!} + \frac{(i\beta t)^4}{4!} + \cdots$$

$$= 1 + i\beta t - \frac{\beta^2 t^2}{2!} - i\frac{\beta^3 t^3}{3!} + \frac{\beta^4 t^4}{4!} + \cdots$$

$$= \left(1 - \frac{\beta^2 t^2}{2!} + \frac{\beta^4 t^4}{4!} - + \cdots\right)$$

$$+ i\left(\beta t - \frac{\beta^3 t^3}{3!} + \frac{\beta^5 t^5}{5!} - + \cdots\right)$$

$$= \cos \beta t + i \sin \beta t.$$

Hence $e^{rt} = e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t)$

$$= e^{\alpha t} \cos \beta t + i\, e^{\alpha t} \sin \beta t.$$

<u>Example</u> $e^{(2+\sqrt{3}i)t} = e^{2t} \cos \sqrt{3}t + i\, e^{2t} \sin \sqrt{3} t.$

~~Theorem~~

<u>Definition</u> Let $f(t)$, $g(t)$ be (real) d'ble functions.

Let $z(t) = f(t) + i\, g(t)$.

$$\frac{dz(t)}{dt} := f'(t) + i\, g'(t).$$

<u>Example</u> Let $z = e^{(2+\sqrt{3}i)t}$.

Find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{d}{dt}\left(e^{2t} \cos \sqrt{3}t\right) + i \frac{d}{dt}\left(e^{2t} \sin \sqrt{3}t\right)$$

$$= e^{2t}(-\sqrt{3} \sin \sqrt{3}t) + 2e^{2t} \cos \sqrt{3}t$$

$$+ i\left(e^{2t}(\sqrt{3} \cos \sqrt{3}t) + 2e^{2t} \sin \sqrt{3}t\right)$$

$= (2+i\sqrt{3})\, e^{2t}\cos\sqrt{3}t \quad + (2i-\sqrt{3})e^{2t}\sin\sqrt{3}t$

$= (2+i\sqrt{3})\, e^{2t}\cos\sqrt{3}t \;+ i(2+i\sqrt{3})e^{2t}\sin\sqrt{3}t$

$= (2+i\sqrt{3})(\,e^{2t}\cos\sqrt{3}t \;+ i\; e^{2t}\sin\sqrt{3}t)$

$= (2+i\sqrt{3}) \quad e^{(2+i\sqrt{3})t}.$

<u>Theorem</u>: Let $r = \alpha + i\beta$ $\quad(\alpha, \beta \in \mathbb{R})$.
Then $\dfrac{d}{dt}\, e^{rt} = r e^{rt}$.

<u>Example</u> Find two independent real solns of
$$y'' - 4y' + 7y = 0.$$

Ans: $r^2 - 4r + 7 = 0$, has roots
$r = 2 \pm \sqrt{3}i$.

Two complex solutions are

$$z_1 = e^{(2+\sqrt{3}i)t}$$
$$= e^{2t}(\cos\sqrt{3}t + i\sin\sqrt{3}t),$$
$$z_2 = e^{(2-\sqrt{3}i)t}$$
$$= e^{2t}(\cos(-\sqrt{3}t) + i\sin(-\sqrt{3}t))$$
$$= e^{2t}(\cos\sqrt{3}t - i\sin\sqrt{3}t)$$

Two real solns are
$$y_1 = \tfrac{1}{2}(z_1 + z_2) = e^{2t}\cos\sqrt{3}t$$
$$y_2 = \tfrac{1}{2i}(z_1 - z_2) = e^{2t}\sin\sqrt{3}t$$

y_1, y_2 are two linearly indept. solns so general soln is

$$y = c_1 e^{2t}\cos\sqrt{3}t + c_2 e^{2t}\sin\sqrt{3}t$$
$$= e^{2t}(c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t))$$

where c_1, c_2 are any constants.

<u>Theorem</u> Suppose $\Delta = b^2 - 4ac < 0$.
The A.E. $ar^2 + br + c = 0$ has two complex solns $r = \alpha \pm i\beta$ where $\alpha = \frac{-b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$.
The DE

$$ay'' + by' + cy = 0$$

has two linearly independent solns

$$y_1 = e^{\alpha t} \cos \beta t, \qquad y_2 = e^{\alpha t} \sin \beta t$$

and the general soln is given by

$$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t),$$

where c_1, c_2 are any constants.