4.5 The Superposition Principle & Undeterminal Coeffs Revisited

Sug. HW odds 1–35, 41

Theorem Suppose y_1, y_2 are two indept. solns of the homogeneous DE

$$(*) \quad ay'' + by' + cy = 0.$$

Suppose y_p is a particular solution to the nonhomog DE

$$(**) \quad ay'' + by' + cy = f(t) \qquad (\text{for } t \in I).$$

Then the general soln to $(**)$ is

$$y = y_p + c_1 y_1 + c_2 y_2,$$

where c_1, c_2 are any constants. Further, the IVP

$$(***) \quad ay'' + by' + cy = f(t), \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1$$

has a unique solution.

Proof in part

$$ay_1'' + by_1' + cy_1 = 0$$
$$ay_2'' + by_2' + cy_2 = 0$$
$$ay_p'' + by_p' + cy_p = f(t).$$

Let $y = y_p + c_1 y_1 + c_2 y_2$ (c_1, c_2 constants).

Then

$$ay'' + by' + cy = a(y_p'' + c_1 y_1'' + c_2 y_2'') + b(y_p' + c_1 y_1' + c_2 y_2') + c(y_p + c_1 y_1 + c_2 y_2)$$

$$= ay_p'' + by_p' + cy_p + c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2)$$

$$= f(t) + c_1 0 + c_2 0 = f(t).$$

Hence $y = y_p + c_1 y_1 + c_2 y_2$ is a soln of $(**)$.

Conversely, suppose $\hat{y}$ is a soln to (**).

Let $y = \hat{y} - y_p$. Then

$$ay'' + by' + cy = a(\hat{y}'' - y_p'') + b(\hat{y}' - y_p') + c(\hat{y} - y_p)$$

$$= a\hat{y}'' + b\hat{y}' + c\hat{y} - (ay_p'' + by_p' + cy_p)$$

$$= f(t) - f(t) = 0.$$

So $\hat{y} - y_p$ is a soln to (*) &

$$\hat{y} - y_p = c_1 y_1 + c_2 y_2 \quad \text{some constants } y_1, y_2,$$

&

$$\hat{y} = y_p + c_1 y_1 + c_2 y_2.$$

Hence the general soln of (**) is

$$y = y_p + c_1 y_1 + c_2 y_2$$

where c_1, c_2 are any constants. □.

Example Find the general soln of

$$y'' + 2y' + y = e^{2x}(9x - 21).$$

We know (from the previous section (see p. 15)) that one particular soln is $y_p = e^{2x}(x-3)$.

A.E: $r^2 + 2r + 1 = 0$

$(r+1) = 0$

$r = -1$ is a double root.

So $y_1 = e^{-x}$, $y_2 = xe^{-x}$ are 2 lin. indept. solns of the homog. DE. Hence the general soln of the nonhom. DE is

$$y = e^{2x}(x-3) + c_1 e^{-x} + c_2 x e^{-x},$$

where c_1, c_2 are any constants.

SUPER POSITION PRINCIPLE

Suppose y_1 is a solution to

(1) $\quad ay'' + by' + cy = g_1(t)$,

and y_2 is a solution to

(2) $\quad ay'' + by' + cy = g_2(t)$.

Then $\quad y = c_1 y_1 + c_2 y_2 \quad$ (c_1, c_2 constant)

is a soln to

(3) $\quad ay'' + by' + cy = c_1 g_1(t) + c_2 g_2(t)$.

Proof:

$$a y_1'' + b y_1' + c y_1 = g_1(t) \quad \text{since } y_1 \text{ is a soln of (1).}$$
$$a y_2'' + b y_2' + c y_2 = g_2(t) \quad \cdots y_2 \cdots\cdots (2).$$

$$a(c_1 y_1 + c_2 y_2)'' + b(c_1 y_1 + c_2 y_2)' + c(c_1 y_1 + c_2 y_2)$$
$$= a(c_1 y_1'' + c_2 y_2'') + b(c_1 y_1' + c_2 y_2') + c(c_1 y_1 + c_2 y_2)$$
$$= c_1(a y_1'' + b y_1' + c y_1)$$
$$\quad + c_2(a y_2'' + b y_2' + c y_2)$$
$$= c_1 g_1(t) + c_2 g_2(t)$$

& $c_1 y_1 + c_2 y_2$ is a soln of (3). $\square$

Example We are given that

$y_1 = \frac{1}{4}\sin 2x \quad$ is a soln to $\quad y'' + 2y' + 4y = \cos 2x$

$y_2 = \frac{x}{4} - \frac{1}{8} \quad$ is a soln to $\quad y'' + 2y' + 4y = x$.

Find a soln to $\quad y'' + 2y' + 4y = 2x - 3\cos 2x$,

$$= -3\cos 2x + 2x.$$

By superposition principle,

$$y = -3y_1 + 2y_2$$
$$= -\tfrac{3}{4}\sin 2x + 2(x/4 - 1/8)$$
$$= -3/4 \sin 2x + x/2 - 1/4$$

is a soln to $\quad y'' + 2y' + 4y = 2x - 3\cos 2x$.

Example (#28) Solve the IVP

(*) $y'' + y' - 12y = e^t + e^{2t} - 1$, $y(0) = 1$, $y'(0) = 3$.

A.E: $r^2 + r - 12 = 0$

$(r-3)(r+4) = 0$

Roots are $r = 3, -4$.

A particular soln of $y'' + y' - 12y = e^t$ has form $y_p = Ae^t$ since $r=1$ not a root of A.E.

A particular soln of $y'' + y' - 12y = e^{2t}$ has form $y_p = Be^{2t}$ since $r=2$ is not a root of A.E.

A particular soln of $y'' + y' - 12y = -1$ has form $y_p = C$ since $-12 \neq 0$.

By superposition, a particular soln of (*) has form

$$y_p = Ae^t + Be^{2t} + C.$$

$$y_p' = Ae^t + 2Be^{2t}$$

$$y_p'' = Ae^t + 4Be^{2t}$$

$$y_p'' + y_p' - 12y_p = -10Ae^t - 6Be^{2t} - 12C = e^t + e^{2t} - 1.$$

$$A = -1/10, \quad B = -1/6, \quad C = +1/12.$$

So

$$y_p = -\frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12}.$$

$r = 3, -4$ so $y_1 = e^{3t}$, $y_2 = e^{-4t}$ are indept. solns of the homog. DE. Hence the general soln of (*) is

$$y = c_1 e^{3t} + c_2 e^{-4t} - \frac{1}{10}e^t - \frac{1}{6}e^{2t} + 1/12.$$

$$y(0) = c_1 + c_2 - 1/10 - 1/6 + 1/12 = 1$$

$$c_1 + c_2 = 1 + 1/10 + 1/6 - 1/12 = \frac{60 + 6 + 10 - 5}{60} = \frac{71}{60}$$

$$y'(t) = 3c_1 e^{3t} - 4c_2 e^{-4t} - 1/10\, e^t - 1/3\, e^{2t}$$

$$y'(0) = 3c_1 - 4c_2 - 1/10 - 1/3 = 3$$

$$c_1 + c_2 = 71/60$$

$$3c_1 - 4c_2 = 3 + 1/10 + 1/3 = \frac{30+3+10}{30} = \frac{103}{30}$$

$$3c_1 + 3c_2 = 71/20 = \frac{213}{60}$$

$$7c_2 = \frac{213}{60} - \frac{103}{30} = \frac{213}{60} - \frac{206}{60} = \frac{7}{60}$$

$$c_2 = 1/60$$

$$c_1 = 70/60 = 7/6.$$

So $$\boxed{y = 7/6\, e^{3t} + 1/60\, e^{-4t} - 1/10\, e^{t} - 1/6\, e^{2t} + 1/12}$$