4.6 Variation of Parameters

Sug Hw: odds 1–17

Variation of parameters is a method for finding a particular soln $y_p(t)$ of a nonhom. linear DE

$$ay'' + by' + cy = g(t),$$

provided we know two linear indept. solns $y_1(t), y_2(t)$ of the corresponding homog. DE

$$ay'' + by' + cy = 0.$$

We assume

$$y_p(t) = v_1(t)\, y_1 + v_2(t)\, y_2$$

$$y_p' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2$$

We assume

$$v_1' y_1 + v_2' y_2 = 0.$$

So

$$y_p' = v_1 y_1' + v_2 y_2',$$

and

$$y_p'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'$$

Therefore,

$$a y_p'' + b y_p' + c y_p = \begin{matrix} c v_1 y_1 & + & c v_2 y_2 \\ + \; b v_1 y_1' & + & b v_2 y_2' \\ + \; a v_1 y_1'' & + & a v_2 y_2'' \end{matrix}$$

$$+ \; a v_1' y_1' + a v_2' y_2'$$

$$= a v_1' y_1' + a v_2' y_2'$$

Since $ay_1'' + by_1' + cy_1 = 0$

& $ay_2'' + by_2' + cy_2 = 0$.

Hence, we want

$$\boxed{\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= \frac{g}{a} \end{aligned}}$$

Solve these equations for v_1', v_2'. Then find v_1, v_2 & y_p.

Example (Final Exam Sp. 2000) (#5)

Find the general soln of the DE

$$y'' - 2y' + y = \frac{e^t}{t}, \quad \text{for } t>0.$$

* First we solve the homog. DE

$$y'' - 2y' + y = 0.$$

A.E: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ is a double root.

Hence $y_1 = e^t$, $y_2 = te^t$ are two lin. indept. solns.

* We use variation of parameters to find a particular soln.

Let

$$y_p = v_1 y_1 + v_2 y_2 = v_1 e^t + v_2 e^t t.$$

We solve

$$y_1 v_1' + y_2 v_2' = 0$$

$$y_1' v_1' + y_2' v_2' = \frac{g}{a} = \frac{e^t}{t}$$

$$e^t v_1' + te^t v_2' = 0$$

$$e^t v_1' + (e^t t + e^t) v_2' = \frac{e^t}{t}$$

$$e^t v_2' = e^t/t, \quad v_2' = 1/t.$$

So $v_2 = \int \frac{1}{t}\, dt$ & $v_2 = \ln t$.

$$v_1' = -tv_2' = -1, \quad \& \quad v_1 = -t.$$

Hence $y_p = -t e^t + (\ln t)\, te^t$,

and the general soln is given by

$$y = c_1 e^t + c_2 te^t - te^t + (\ln t)\, te^t,$$

for c_1, c_2 any constants.