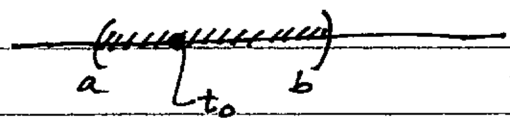
4.7 Variable Coefficient Equations

Existence-Uniqueness Theorem for 2nd order Linear IVP

Suppose $p(t)$, $q(t)$, $g(t)$ are continuous on an open interval (a,b) which contains the point t_0



Let Y_0, Y_1 be constants. The Initial Value Problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = Y_0,\ y'(t_0) = Y_1$$

has a unique solution $y(t)$ valid on the interval (a,b) ie valid for $a<t<b$.

Example (#4) Discuss the existence & uniqueness of a solution to the IVP

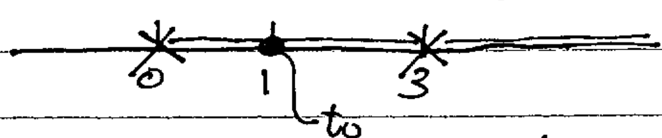
$$e^t y'' - \frac{y'}{(t-3)} + y = \ln t, \qquad y(1) = Y_0,\ y'(1) = Y_1$$

$$\Leftrightarrow \quad y'' + \left(\frac{-e^{-t}}{t-3}\right)y' + e^{-t} = e^{-t}\ln t.$$

$p(t) = \dfrac{-e^{-t}}{t-3}$ is continuous for $t \neq 3$.

$q(t) = e^{-t}$ is continuous for all t.

$g(t) = e^{-t}\ln t$ is continuous for all $t>0$.



$p(t)$, $q(t)$, $g(t)$ are continuous on the open interval $(0,3)$ which ~~exists~~ contains the point $t_0 = 1$.

The interval $(0,3)$ is the largest open interval which contains the point $t_0 = 1$ & on which all three functions $p(t)$, $q(t)$, and $g(t)$ are continuous.
The Existence-Uniqueness Theorem implies that the IVP has a unique solution $y(t)$ valid on the interval $(0,3)$.

Example (#35)
Given that $1+t$, $1+2t$, $1+3t^2$ are solutions to the DE

$$y'' + p(t)y' + q(t)y = g(t)$$

find the solution that satisfies $y(1) = 2$, $y'(1) = 0$.
Let $L[y] = y'' + p(t)y' + q(t)y$.
Let $z_1 = 1+t$, $z_2 = 1+2t$, $z_3 = 1+3t^2$.
Then $L[z_1] = g(t)$

$L[z_2] = g(t)$

$L[z_3] = g(t)$.

$L[z_1 - z_2] = L[z_1] - L[z_2] = g(t) - g(t) = 0$.

So $y_1 = z_1 - z_2 = -t$ is a soln to the homog. eqn.

$L[z_1 - z_3] = L[z_1] - L[z_3] = g(t) - g(t) = 0$.

So $y_2 = z_1 - z_3 = (1+t) - (1+3t^2) = t - 3t^2$

is a solution to the homog. eqn.

y_1, y_2 are linearly indept. since

$$W[y_1, y_2] = \begin{vmatrix} -t & t-3t^2 \\ -1 & 1-6t \end{vmatrix}$$

$$= -t(1-6t) + (t-3t^2) = 3t^2 \neq 0$$

for $t \neq 0$.

$y_p = 1 + t$ is a particular soln. So the general soln of the DE is given by

$$y = y_p + c_1 y_1 + c_2 y_2$$

$$y = (1+t) + c_1(-t) + c_2(t - 3t^2)$$

for any constants c_1, c_2.

Whence $y' = 1 - c_1 + c_2(1 - 6t)$

We want

$$y(1) = 2 - c_1 - 2c_2 = 2$$

$$y'(1) = 1 - c_1 - 5c_2 = 0$$

$$c_1 + 2c_2 = 0$$

$$c_1 + 5c_2 = +1$$

$$3c_2 = +1$$

$$c_2 = +1/3$$

$$c_1 = -2c_2 = -2/3.$$

So $y = (1+t) + \frac{2}{3}t + \frac{1}{3}(t - 3t^2)$

$= 1 + t + \frac{2}{3}t + \frac{1}{3}t - t^2$

$= 1 + 2t - t^2$

is the soln to the IVP.

Cauchy-Euler Equations:

$$at^2y''(t) + bty'(t) + cy = f(t),$$

where a, b, c are constants, $a \neq 0$, and $t > 0$.

To solve the homogeneous Cauchy-Euler Equation

$$at^2y'' + bty' + cy = 0$$

we try

$$y = t^r \quad \text{where } r \text{ is a constant.}$$

Then $y' = rt^{r-1}$, $ty' = rt^r$,

$y'' = r(r-1)t^{r-2}$, $t^2y'' = r(r-1)t^r$,

and

$$at^2y'' + bty' + cy = ar(r-1)t^r + brt^r + ct^r$$

$$= t^r\left(ar(r-1) + br + c\right) = 0$$

if

$$ar(r-1) + br + c = 0 \quad \text{(Characteristic Equation)}$$

THEOREM If

$$ar(r-1) + br + c = 0$$

then $y = t^r$ is a solution to the Homogeneous Cauchy-Euler Equation

$$at^2y'' + bty + c = 0.$$

(#10) **Example** Find the general solution to

$$t^2y'' + 7ty' - 7y = 0 \quad \text{for } t > 0.$$

Characteristic Eqn:

$$r(r-1) + 7r - 7 = 0$$

$$r^2 + 6r - 7 = 0$$

$$(r-1)(r+7)=0$$
$$r=1,-7.$$

So $y_1=t^1$, $y_2=t^{-7}$ are two linearly independent solutions. Hence the general solution is given by
$$y=c_1t+c_2t^{-7}$$
where c_1, c_2 are any constants.

THEOREM Suppose the characteristic Eqn
$$ar(r-1)+br+c=0$$
has two distinct real roots $r=r_1, r_2$.
Then $y_1=t^{r_1}$, $y_2=t^{r_2}$ are linearly indept. solutions of $at^2y''+bty'+cy=0$ $(t>0)$.

Example Solve
$$t^2y''-ty'+y=0 \quad \text{where } t>0.$$
Characteristic Eqn:
$$r(r-1)-r+1=0$$
$$r^2-2r+1=0$$
$$(r-1)^2=1$$
$r=1$ is a double root.

So $y_1=t$ is a solution.
Let $y_2=t\,v(t)$. Then
$$y_2'=tv'+v$$
$$y_2''=tv''+v'+v' = tv''+2v'$$
$$\cancel{t^2y_2''-ty_2'+y_2=t^2(tv''+(t+1)v''+v')}$$
$$\cancel{-t(tv'+v)+tv}$$

$$t^2 y_2'' - t y_2' + y_2 = t^2(tv'' + 2v') - t(tv' + v) + tv$$
$$= t^3 v'' + 2t^2 v' - t^2 v'$$
$$= t^3 v'' + t^2 v' = 0$$
$$v'' + \frac{1}{t} v' = 0 \qquad \text{(assuming } t > 0\text{).}$$

$$\mu(t) = \exp\left(\int \frac{1}{t}\,dt\right) = t$$

$$\frac{d}{dt}(tv') = tv'' + v' = 0$$
$$tv' = c = 1$$
$$v' = 1/t$$
$$v = \ln t$$

So $y_2 = (\ln t)\,t$ is a also a soln.

Thus

$y_1 = t, \quad y_2 = t \ln t$ are two

lin indept. solns & the general solution is given by

$$y = c_1 t + c_2 t \ln t \qquad (t > 0)$$

where c_1, c_2 are any constants.

<u>DOUBLE ROOT</u> If $r = r_1$ is a double root of

$$ar(r-1) + br + c = 0$$

then $y_1 = t^{r_1}, \quad y_2 = t^{r_1} \ln t$ are

linearly indept. solutions of the Cauchy-Euler Eqn

$$at^2 y'' + bty' + cy = 0.$$

COMPLEX ROOTS

Recall

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (\underline{\text{EULER}})$$

for real θ.

Suppose $r = \alpha \pm i\beta$ $(\beta > 0)$ are complex roots of the Char. Eqn: $a\,r(r-1) + br + c = 0$.

$$z = t^r = t^{\alpha + i\beta} = t^{\alpha}\, t^{i\beta}$$

$$= t^{\alpha}\left(\left(e^{\ln t}\right)^{i\beta}\right) = t^{\alpha} e^{i\beta \ln t}$$

$$= t^{\alpha}\left(\cos(\beta \ln t) + i \sin(\beta \ln t)\right)$$

is a complex sol$^{\underline{n}}$.

$$y_1 = t^{\alpha}\cos(\beta \ln t), \quad y_2 = t^{\alpha}\sin(\beta \ln t)$$

are linearly indept. solns of the Cauchy-Euler Eqn

$$a t^2 y'' + b t y' + c y = 0.$$

Example (#17)

Solve

$$t^2 y'' + 5ty' + 4y = 0 \qquad (t > 0)$$

Char. Eqn:

$$r(r-1) + 5r + 4 = 0$$

$$r^2 + 4r + 4 = 0$$

$$r^2 + 4r = -4$$

<u>Example</u> Solve

$$t^2y'' - ty' + 2y = 0.$$

Char. Eqn:

$$r(r-1) - r + 2 = 0$$
$$r^2 - 2r - 2 = 0$$
$$r^2 - 2r = -2$$
$$r^2 - 2r + 1 = -1$$
$$(r-1)^2 = -1$$
$$r - 1 = \pm i$$
$$r = 1 \pm i$$

$\alpha = \beta = 1.$

$$y_1 = t\cos(\ln t), \quad y_2 = t\sin(\ln t)$$

are linearly indept. solutions & the general soln is given by

$$y = t\,(c_1\cos(\ln t) + c_2\sin(\ln t))$$

where c_1, c_2 are any constants.