Damped Case There is a force proportional to $y'(t)$

$$F_{damping} = -b\,y'(t)$$

$$F = ma = -ky - by'(t)$$

$$my''(t) = -ky - by'$$

$$my'' + by' + ky = 0.$$

A.E: $mr^2 + br + k = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Underdamped (or oscillatory) motion

This occurs when $b^2 < 4mk$ and A.E. has complex conjugate roots $r = \alpha \pm i\beta$

where $\alpha = -\frac{b}{2m}$, $\beta = \frac{\sqrt{4mk - b^2}}{2m}$.

The general soln has the form

$$y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$$

$$= e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

$$= A e^{\alpha t}\sin(\beta t + \phi)$$

where $A = \sqrt{c_1^2 + c_2^2}$, $\tan\phi = c_1/c_2$.