Resonance

Undamped system $(b=0)$

Consider a system

$$my'' + ky = F_0 \cos \gamma t$$

A.E: $\quad mr^2 + k = 0$

$$mr^2 = -k$$

$$r^2 = -\frac{k}{m}, \quad r = \pm i\sqrt{\frac{k}{m}}.$$

Note: If $\gamma \neq \sqrt{k/m}$ then

$$y_p = A\cos\gamma t + B\sin\gamma t$$

The system is at resonance if $\gamma = \sqrt{\frac{k}{m}}$.

Let $\omega = \sqrt{\frac{k}{m}}$ and suppose $\gamma = \omega$. Then

$$y_p = t(A\cos\omega t + B\sin\omega t) = t\,y_h$$

$$\cancel{y_p' = t(-A\omega\sin\omega t + B\omega\cos\omega t) + (A\cos\omega t + B\sin\omega t)}$$

$$\cancel{y_p'' =}$$

$$y_p' = t\,y_h' + y_h$$

$$y_p'' = t\,y_h'' + y_h' + y_h' = t\,y_h'' + 2y_h'$$

$$\begin{aligned} my_p'' + ky_p &= m(t\,y_h'' + 2y_h') \\ &\quad + k\,t\,y_h \\ &= t(my_h'' + ky_h) + 2my_h' \\ &= 2my_h' \end{aligned}$$

$$2my_h' = 2m(-A\sin\omega t + B\omega\cos\omega t) = F_0\cos\omega t$$

$$2Bm\omega\cos\omega t - 2Am\sin\omega t = F_0\cos\omega t$$

So $A = 0$ & $2Bm\omega = F_0$,

$$B = \frac{F_0}{2m\omega}$$

$$y_p(t) = \frac{F_0}{2m\omega}\, t \sin\omega t$$