6.2 Homogeneous Linear Equations with Constant Coefficients

PRODUCT OF TWO OPERATORS

Suppose $L_1 : A \to B$ and $L_2 : B \to C$ are transformations. The product $L_2 L_1$ is defined by

$$(L_2 L_1)(a) = L_2(L_1(a)) \quad \text{for } a \in A.$$

Example $(D+3)(y) = \frac{dy}{dx} + 3y$

$$(D+2)(y) = \frac{dy}{dx} + 2y.$$

Show that $(D+2)(D+3) = (D+3)(D+2) = D^2 + 5D + 6.$

$$((D+2)(D+3))\,y = (D+2)\left(\frac{dy}{dx} + 3y\right)$$

$$= D\left(\frac{dy}{dx} + 3y\right) + 2\left(\frac{dy}{dx} + 3y\right)$$

$$= y''(x) + 3\frac{dy}{dx} + 2\frac{dy}{dx} + 6y$$

$$= y''(x) + 5\frac{dy}{dx} + 6y$$

$$= D(D(y)) + 5D(y) + 6(y)$$

$$= (D^2 + 5D + 6)(y).$$

Hence $(D+2)(D+3) = D^2 + 5D + 6.$

Similarly $(D+3)(D+2) = D^2 + 5D + 6.$

NOTE ① $D(y) = \frac{dy}{dx}$ ② $D^2(y) = y''(x).$

③ $D = \frac{d}{dx}$, ④ $D^2 = \frac{d^2}{dx^2}$

Proposition: Suppose a, b are constants.
Then $(D+a)(D+b) = (D+b)(D+a)$
$= D^2 + (a+b)D + ab.$

Example: Let $D = \frac{d}{dx}$.
Show that
$$D(D+x) \neq (D+x)D.$$

$$D(D+x)y = D((D+x)y)$$
$$= D\left(\frac{dy}{dx} + xy\right)$$
$$= D\left(\frac{dy}{dx}\right) + D(xy)$$
$$= y''(x) + x\frac{dy}{dx} + y$$
$$= (D^2 + xD + 1)y$$

$$(D+x)Dy = (D+x)\left(\frac{dy}{dx}\right)$$
$$= D\left(\frac{dy}{dx}\right) + x\frac{dy}{dx} = y'' + x\frac{dy}{dx}$$
$$= (D^2 + xD)y$$

& we see that
$$D(D+x) \neq (D+x)D.$$

__Example__: Show that

$$y_1(x) = e^{x}, \quad y_2(x) = e^{2x}, \quad y_3(x) = e^{3x}$$

are linearly independent on $(-\infty, \infty)$ using Differential Operators.

Suppose $\quad c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = 0$ for all x.

$$(D-1)\, y_1 = e^{x} - e^{x} = 0.$$
$$(D-2)\, y_2 = 2e^{2x} - 2e^{2x} = 0$$
$$(D-3)\, y_3 = 3e^{3x} - 3e^{3x} = 0.$$

$\cancel{(D-3)(D-2)}$

We want an operator that annihilates both y_2 & y_3, & does not annihilate y_1.

$$(D-3)(D-2)\, y_2 = (D-3)\, 0 = 0.$$
$$(D-3)(D-2)\, y_3 = (D-2)(D-3)\, y_3 = (D-2)\, 0 = 0.$$

$(D-3)(D-2)\, y_1 \neq 0$ since $y_1 = e^{x}$ is not a solution of

$$y'' - 5y' + 6y = 0.$$

$$(D-3)(D-2)(c_1 y_1 + c_2 y_2 + c_3 y_3) = 0$$

$$c_1 (D-3)(D-2) y_1 + c_2 (D-3)(D-2) y_2 + c_3 (D-3)(D-2) y_1 = 0$$
$$c_1 \big((D-3)(D-2) y_1\big) = 0$$

& Hence $c_1 = 0$.

Similarly we can show that $c_2 = 0$ & $c_3 = 0$ using the operators $(D-1)(D-3)$ & $(D-1)(D-2)$.

A general n-th order linear homogeneous DE with constant coefficients has the form

$(*)$ $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$

where $a_0, a_1, \ldots, a_n$ are constants.

$$(*) \Leftrightarrow \left(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\right) y = 0, \quad \text{where } D = \frac{d}{dx}.$$

$y = e^{rx}$ is a soln of $(*)$ iff
$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$

CASE 1 Distinct Real Roots

Suppose $r_1, r_2, \ldots, r_n$ are real & distinct solns of $P(r) = 0$.
Then
$$P(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n)$$
&
$$(*) \Leftrightarrow a_n (D - r_1)(D - r_2) \cdots (D - r_n) y = 0$$
$y_1(x) = e^{r_1 x}$, $y_2 = e^{r_2 x}, \ldots, y_n = e^{r_n x}$ are n linearly independent solns of $(*)$ & the general soln of $(*)$ is
$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$
where $c_1, c_2, \ldots, c_n$ are any constants.

CASE 2 Complex Roots

Suppose $r = \alpha + i\beta$ $(\beta \neq 0)$ is a complex soln of $P(r) = 0$.
Then $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$ are independent solns of $(*)$