CASE 3A Repeated Roots (real)

Suppose $r = r_1$ is a root of multiplicity m of $P(r) = 0$.
This means $P(r) = a_n (r - r_1)^m Q(r)$ $(\& \; Q(r_1) \neq 0)$.
also $P(D) = a_n D^n + \cdots + a_0 = a_n (D - r_1)^m Q(D)$.
Then $y_1 = e^{r_1 x}$, $y_2 = x e^{r_1 x}$, $\ldots$, $y_m = x^{m-1} e^{r_1 x}$
are indept. solns of $(*)$

CASE 3B Repeated Complex (Roots).

Suppose $r = \alpha \pm i\beta$ are complex roots of multiplicity m of $P(r) = 0$.
This means that

$$P(r) = a_n (r - (\alpha + i\beta))^m (r - (\alpha - i\beta))^m \, Q(r) \quad , \; Q(\alpha \pm i\beta) \neq 0.$$

$$= a_n (r^2 + Ar + B)^m Q(r)$$

where $r = \alpha \pm i\beta$ are complex roots of $(r^2 + Ar + B) = 0$.

Then

$$y_1 = e^{\alpha x} \cos\beta x, \quad y_2 = x e^{\alpha x} \cos\beta x, \ldots, \quad y_m = x^{m-1} e^{\alpha x} \cos\beta x$$

$$z_1 = e^{\alpha x} \sin\beta x, \quad z_2 = x e^{\alpha x} \sin\beta x, \ldots, \quad z_m = x^{m-1} e^{\alpha x} \sin\beta x$$

are linearly indept. solns of $(*)$.

#16 (p. 356) Find the general solution of

$$(*) \quad (D+1)^2(D-6)^3(D+5)(D^2+1)(D^2+4)[y] = 0.$$

This is a linear homogeneous 10^{th} DE with constant coefficients.

Linearly independent solns of $(D+1)^2[y]=0$ are
$y_1 = e^{-x}, \quad y_2 = xe^{-x}$.

Linearly independent solns of $(D-6)^3[y]=0$ are
$y_3 = e^{6x}, \quad y_4 = xe^{6x}, \quad y_5 = x^2e^{6x}$.

Linearly independent solns of $(D+5)[y]=0$ is
$y_6 = e^{-5x}$.

Linearly independent solns of $(D^2+1)[y]=0$
are $y_7 = \cos x, \quad y_8 = \sin x$ since roots of $r^2+1=0$ are $r = \pm i$.

Linearly indept. solns of $(D^2+4)[y]=0$
are $y_9 = \cos 2x, \quad y_{10} = \sin 2x$ since roots of $r^2+4=0$ are $r = \pm 2i$.

The functions $y_1, y_2, \ldots, y_{10}$ are linearly indept. solns of $(*)$.
Hence the general soln of $(*)$ is given by

$$y = c_1 e^{-x} + c_2 xe^{-x} + c_3 e^{6x} + c_4 xe^{6x} + c_5 x^2 e^{6x} + c_6 e^{-5x} + c_7 \cos x + c_8 \sin x + c_9 \cos 2x + c_{10} \sin 2x,$$

where $c_1, c_2, \ldots, c_{10}$ are any constants.