Example Solve $y''' + y'' - y' - y = e^x \quad (*)$

~~Solve $y''' + y'' - y' - y = e^x (*)$~~

$$\Leftrightarrow \quad (D^3 + D^2 - D - 1)[y] = e^x.$$

$$\begin{aligned} D^3 + D^2 - D - 1 &= D^2(D+1) - (D+1) \\ &= (D^2 - 1)(D+1) = (D-1)(D+1)^2 \end{aligned}$$

$$(*) \Leftrightarrow \quad (D-1)(D+1)^2 [y] = e^x.$$

$D - 1$ annihilates e^x.

So if y is a soln to $(*)$ then

$$(D-1)^2 (D+1)^2 [y] = 0,$$

&

$$y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}$$

for constants c_1, c_2, c_3, c_4.

We calculate $(D-1)(D+1)^2 [y]$.

$(D-1)(D+1)^2$ annihilates $c_1 e^x + c_3 e^{-x} + c_4 x e^{-x}$.

So

$$(D-1)(D+1)^2 [y] = (D-1)(D+1)^2 (c_2 x e^x)$$

<u>NOTE</u>: $(D-1)^2 (c_2 x e^x) = 0$

So

$$\begin{aligned} (D-1)(D+1)^2 &= (D-1)(D-1+2)^2 \\ &= (D-1)\left((D-1)^2 + 2(D-1) + 4\right) \\ &= (D-1)^3 + 2(D-1)^2 + 4(D-1). \end{aligned}$$

Hence,

$$(D-1)(D+1)^2 (c_2 x e^x)$$

$$= 4c_2 (D-1)(x e^x)$$
$$= 4c_2 (-x e^x + x e^x + e^x)$$
$$= 4c_2 e^x,$$

&

$$(D-1)(D+1)^2 [y] = 4c_2 e^x = e^x$$ &

we require $c_2 = 1/4$.

The general soln is given by

$$y = c_1 e^x + 1/4 \, x e^x + c_3 e^{-x} + c_4 x e^{-x}$$

where c_1, c_3, c_4 are any constants.