Example Let $a \geq 0$ be a constant. Find $\mathcal{L}\{e^{at}\}$.

We know $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = 1/s$ for $s > 0$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \frac{1}{s-a} \quad \text{if } s-a > 0 \text{, ie } s > a.$$

Hence
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \text{ for } s > a.$$

Example Find $\mathcal{L}\{\sin bt\}$, $\mathcal{L}\{\cos bt\}$ where b is a constant.

$$e^{ibt} = \cos bt + i \sin bt \quad \text{(Euler)}.$$

$$\mathcal{L}\{e^{ibt}\} = \frac{1}{s-ib} \quad \text{(proof omitted)}$$

$$= \frac{s+ib}{(s-ib)(s+ib)} = \frac{s+ib}{s^2+b^2}$$

$$= \frac{s}{s^2+b^2} + i\,\frac{b}{s^2+b^2}.$$

But
$$\mathcal{L}\{e^{ibt}\} = \int_0^\infty e^{-st} \cdot e^{ibt}\,dt$$

$$= \int_0^\infty e^{-st}(\cos bt + i \sin bt)\,dt$$

$$= \int_0^\infty e^{-st} \cos bt + i\, e^{-st} \sin bt\,dt$$

$$= \mathcal{L}\{\cos bt\} + i\,\mathcal{L}\{\sin bt\}.$$