Chapter 7 Laplace Transforms

7.2 Definition of Laplace Transform (Sug Hw: odds 1–29)

Let $f(t)$ be a function defined for $t \geq 0$ (ie on the interval $[0,\infty)$). The Laplace transform of f denoted by $\mathcal{L}\{f\}$ is the function

$$F(s) := \int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt$$

provided the limit exists.

We write $\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt.$

Example Find $\mathcal{L}\{1\}$ using the definition.

$$\int_0^N e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right]_0^N \qquad (\text{if } s \neq 0)$$

$$= \frac{e^{-sN}}{-s} + \frac{e^0}{s} = \frac{1}{s} - \frac{e^{-sN}}{s} = \frac{1}{s} - \frac{1}{s e^{sN}}.$$

$$\lim_{N\to\infty} \int_0^N e^{-st}\,dt = \lim_{N\to\infty} \frac{1}{s} - \frac{1}{s e^{sN}}$$

$$= \frac{1}{s} - 0 \qquad (\text{provided } s>0).$$

$$= \frac{1}{s}.$$ So $\int_0^\infty e^{-st}\cdot 1\,dt = \mathcal{L}\{1\} = 1/s$ for $s>0$.

Hence

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{for } s>0.$$

<u>Example</u> Let $a>0$ be a constant. Find $\mathcal{L}\{e^{at}\}$.

We know $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = 1/s$ for $s>0$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \frac{1}{s-a} \quad \text{if } s-a>0 \text{, ie } s>a.$$

Hence
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{for } s>a.$$

<u>Example</u> Find $\mathcal{L}\{\sin bt\}$, $\mathcal{L}\{\cos bt\}$ where b is a constant.

$$e^{ibt} = \cos bt + i\sin bt \qquad \text{(Euler).}$$

$$\mathcal{L}\{e^{ibt}\} = \frac{1}{s-ib} \qquad \text{(proof omitted)}$$

$$= \frac{s+ib}{(s-ib)(s+ib)} = \frac{s+ib}{s^2+b^2}$$

$$= \frac{s}{s^2+b^2} + i\,\frac{b}{s^2+b^2}.$$

But
$$\mathcal{L}\{e^{ibt}\} = \int_0^\infty e^{-st} \cdot e^{ibt}\,dt$$

$$= \int_0^\infty e^{-st}(\cos bt + i\sin bt)\,dt$$

$$= \int_0^\infty e^{-st}\cos bt + i\,e^{-st}\sin bt\,dt$$

$$= \mathcal{L}\{\cos bt\} + i\,\mathcal{L}\{\sin bt\}.$$

Therefore,

$$\mathcal{L}\{\cos bt\} + i\mathcal{L}\{\sin bt\} = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}.$$

It follows that

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}, \quad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}.$$

It can be shown that this is valid for $s>0$.

Linearity of Laplace Transform

(1) $\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$

(2) $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ if c is a constant.

(3) $\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}$

if c_1, c_2 are constants.

Example (#14) Using Laplace transform table & linearity find

$$\mathcal{L}\{5 - e^{2t} + 6t^2\}.$$

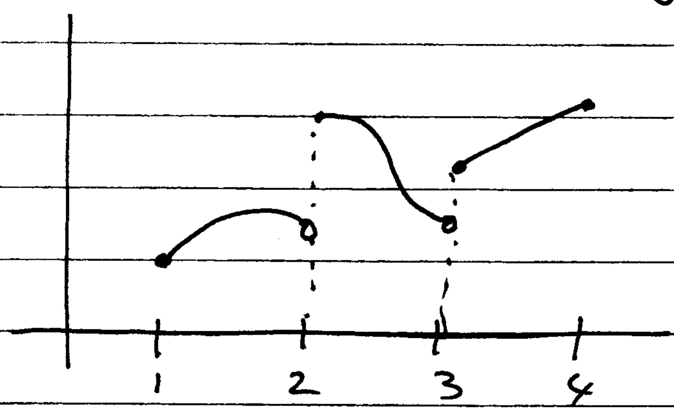
$\mathcal{L}\{5 - e^{2t} + 6t^2\} = \mathcal{L}\{5\} - \mathcal{L}\{e^{2t}\} + 6\mathcal{L}\{t^2\}$ (by linearity)

$= 5\mathcal{L}\{1\} - \mathcal{L}\{e^{2t}\} + 6\mathcal{L}\{t^2\}$

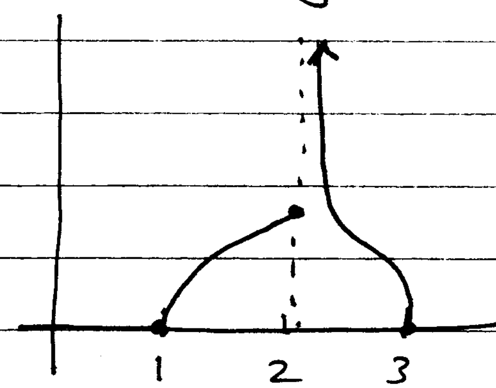
$= \dfrac{5}{s} - \dfrac{1}{s-2} + 6 \cdot \dfrac{2!}{s^3}$ (by 1., 2., 3. in table)

$= \dfrac{5}{s} - \dfrac{1}{s-2} + \dfrac{12}{s^3}$ for $s>2$.

Definition A function $f(t)$ is *piecewise continuous* on a finite interval $[a,b]$ if it is continuous at every point in $[a,b]$ except for possibly a finite number of points at which $f(t)$ has a jump discontinuity.



piecewise CTS



not piecewise CTS

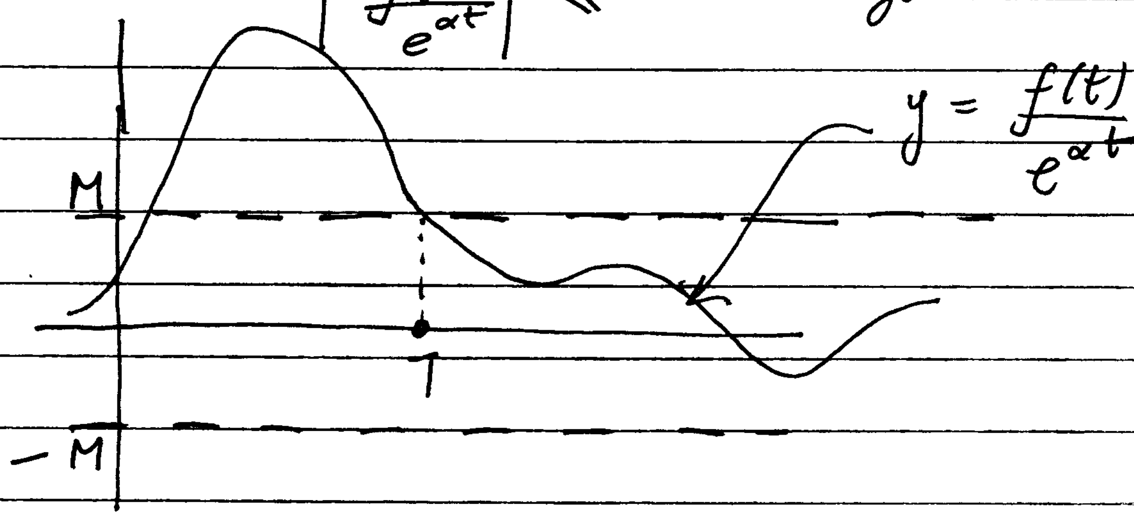
$f(t)$ is *piecewise continuous* on $[0,\infty)$ if it is piecewise continuous on $[0,N]$ for each $N>0$.

Definition A function $f(t)$ is of *exponential order* α if there are constants M, T:

$$|f(t)| \leq M e^{\alpha t} \quad \text{for all } t \geq T$$

or

$$\left| \frac{f(t)}{e^{\alpha t}} \right| \leq M \quad \text{for all } t \geq T.$$



<u>Example</u> Determine whether $f(t) = t\cos t$ has exponential ~~and~~ order.

$$|f(t)| = |t\cos t| = |t|\,|\cos t| \leq |t|$$

for all t, since $0 \leq |\cos t| \leq 1$.

$$\lim_{t\to\infty} \frac{t}{e^t} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{t\to\infty} \frac{1}{e^t} \quad \text{(by L'H's rule)}$$

$$= 0.$$

Hence $\left|\frac{t}{e^t}\right| \leq 1$ eventually,

and there is a $T > 0$:

$$|t| \leq e^t \quad \text{for } t \geq T$$

& $\quad |f(t)| \leq |t| \leq e^t \quad$ for $t \geq T$.

So $f(t)$ has exponential order $\alpha = 1$.

[Note: It can be shown that $t \leq e^t$ for all t.
See Bonus Prob. 4. $f(t) = t\cos t$ has exponential order α for any $\alpha > 0$.]

<u>Example</u> Determine whether $f(t) = e^{t^2}$ has exponential order. Let α be a constant.

$$~~\text{Now}~~ \lim_{t\to\infty} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t\to\infty} e^{t^2-\alpha t} = +\infty$$

since $\lim_{t\to\infty} t^2 - \alpha t = \lim_{t\to\infty} t(t-\alpha) = +\infty$.

Hence $f(t)$ can not be of expential order α, for any α. So $f(t)$ is not of exponential order.