$$Y(s) = \frac{s\,y(0) + y'(0)}{s^2 + b^2}.$$

Two solutions of the DE are $y_1 = \cos bt$, $y_2 = \sin bt$.

Hence

$$\mathcal{L}\{\cos bt\} = \frac{s\,y_1(0) + y_1'(0)}{s^2 + b^2} = \frac{s}{s^2+b^2}$$

$$\mathcal{L}\{\sin bt\} = \frac{s\,y_2(0) + y_2'(0)}{s^2 + b^2} = \frac{b}{s^2+b^2}.$$

<u>Theorem</u> Suppose $f(t)$ is piecewise CTS on $[0, \infty)$ & of exponential ~~for~~ order α. Let $F(s) = \mathcal{L}\{f\}$.
Then $\mathcal{L}\{t\,f(t)\} = -F'(s)$ for $s > \alpha$.

<u>Proof</u>

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right)dt \qquad \text{(by a Thm from adv. Calc.)}$$

$$= \int_0^\infty -t\,e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\left(t\,f(t)\right)dt$$

$$= -\mathcal{L}\{t\,f(t)\}.$$

So $\mathcal{L}\{t\,f(t)\} = -F'(s)$. $\square$