7.4 The Inverse Laplace Transform

Sug. HW: odds 1–29

Question What function $f(t)$ has Laplace transform $\frac{1}{(s-2)^2}$?

Table (4. $k=2, n=1$) gives

$$\mathcal{L}\{e^{2t}t\} = \frac{1!}{(s-2)^2} = \frac{1}{(s-2)^2}.$$

So $f(t) = e^{2t}t$ has Laplace transform $\frac{1}{(s-2)^2}$.

$$\mathcal{L}\{e^{2t}t\} = \frac{1}{(s-2)^2}.$$

We write $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = te^{2t}$.

It can be shown that there is no other continuous function whose Laplace transform is te^{2t}.

Question What function $f(t)$ has Laplace transform $\frac{-s}{(s^2+4)^2}$?

$$\mathcal{L}\{t\sin 2t\} = \frac{4s}{(s^2+4)^2} \quad \text{for } s>0 \text{ (Table (14))}.$$

$$\mathcal{L}\left\{-\frac{1}{4}t\sin 2t\right\} = -\frac{1}{4}\left(\frac{4s}{(s^2+4)^2}\right) = \frac{-s}{(s^2+4)^2}.$$

So $\mathcal{L}^{-1}\left\{\frac{-s}{(s^2+4)^2}\right\} = -\frac{t}{4}\sin 2t.$

Definition Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies

$$\mathcal{L}\{f\} = F(s)$$

Then we say $f(t)$ is the inverse Laplace transform of $F(s)$ and write: $f = \mathcal{L}^{-1}\{F\}$.

Linearity of inverse transform

Suppose $\mathcal{L}^{-1}\{F\}$, $\mathcal{L}^{-1}\{f_1\}$, $\mathcal{L}^{-1}\{f_2\}$ exist and continuous on $[0,\infty)$. Let c, c_1, c_2 be any constants. Then

(1) $\mathcal{L}^{-1}\{f_1 + f_2\} = \mathcal{L}^{-1}\{f_1\} + \mathcal{L}^{-1}\{f_2\}$

(2) $\mathcal{L}^{-1}\{cf\} = c\,\mathcal{L}^{-1}\{f\}$

(3) $\mathcal{L}^{-1}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}^{-1}\{f_1\} + c_2\mathcal{L}^{-1}\{f_2\}$.

Example (#6) Find $\mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3}\right\}$.

We use table #4: $\mathcal{L}\{e^{kt}t^n\} = \frac{n!}{(s-k)^{n+1}}$.

$$\frac{3}{(2s+5)^3} = \frac{3}{[2(s+5/2)]^3} = \frac{3}{8(s+5/2)^3}$$

Let $k = -5/2$, $n = 2$.

$$\mathcal{L}\{e^{-5t/2}t^2\} = \frac{2!}{(s+5/2)^3}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{8(s+5/2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{16}\,\frac{2!}{(s+5/2)^3}\right\}$$

$$= 3/16\,\mathcal{L}^{-1}\left\{\frac{2!}{(s+5/2)^3}\right\} = 3/16\,e^{-5t/2}t^2.$$

Example: Let $Y = \dfrac{s-1}{2s^2+8s+26}$. Find $\mathcal{L}^{-1}\{Y\}$.

$$Y = \frac{s-1}{2s^2+8s+26} = \frac{s-1}{2(s^2+4s+13)} = \frac{s-1}{2((s+2)^2+9)}$$

$$= \frac{s-1}{2((s+2)^2+3^2)} = \frac{(s+2)-3}{2((s+2)^2+3^2)}$$

$$= \frac{s+2}{2((s+2)^2+3^2)} - \frac{3}{2((s+2)^2+3^2)}$$

$$\mathcal{L}\{e^{-2t}\cos 3t\} = \frac{s+2}{(s+2)^2+3^2} \qquad (\text{\# 9 in Table})$$

$$\mathcal{L}\{e^{-2t}\sin 3t\} = \frac{3}{(s+2)^2+3^2} \qquad (\text{\# 8 in Table}).$$

$$\mathcal{L}^{-1}\{Y\} = \tfrac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} - \tfrac{1}{2}\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}$$

$$= \tfrac{1}{2}e^{-2t}\cos 3t - \tfrac{1}{2}e^{-2t}\sin 3t.$$

Let $R(s) = \frac{P(s)}{Q(s)}$ be a rational function of s where $\deg P < \deg Q$. We can compute $\mathcal{L}^{-1}\{R(s)\}$ using the method of partial fractions.

<u>Case 1</u> <u>$Q(s)$ has only nonrepeated linear factors.</u>

<u>Example</u> Find $\mathcal{L}^{-1}\left\{\frac{s+11}{(s-1)(s+3)}\right\}$.

$$\frac{s+11}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}.$$

$$s+11 = A(s+3) + B(s-1)$$

$s=1$: $\quad 12 = 4A, \quad A = 3.$

$s=-3$ $\quad 8 = -4B, \quad B = -2.$ So

$$\frac{s+11}{(s-1)(s+3)} = \frac{3}{s-1} - \frac{2}{s+3}$$

$$\mathcal{L}^{-1}\left\{\frac{s+11}{(s-1)(s+3)}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= 3e^{t} - 2e^{-3t}.$$

In general, if $Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n)$ where $r_1, r_2, \ldots, r_n$ are all different then

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \cdots + \frac{A_n}{s-r_n}$$

for some constants $A_1, A_2, \ldots, A_n$ when $\deg P < \deg Q$.

Case 2 Repeated Linear Factors

Example Find $\mathcal{L}^{-1}\left\{ \frac{4s^2-11s+12}{(s-2)^2(s+1)} \right\}$.

$$\frac{4s^2-11s+12}{(s-2)^2(s+1)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s+1}$$

$$4s^2-11s+12 = A(s-2)(s+1)+B(s+1)+C(s-2)^2.$$

$s=2 \Rightarrow \quad 16-22+12 = 3B, \quad 3B=6, \quad B=2.$

$s=-1 \Rightarrow \quad 4+11+12 = 9C, \quad 9C=27, \quad C=3.$

$s=0 \Rightarrow \quad 12 = -2A+B+4C = -2A+2+12,$

$\quad -2A=-2, \quad A=1.$

So

$$\frac{4s^2-11s+12}{(s-2)^2(s+1)} = \frac{1}{s-2} + \frac{2}{(s-2)^2} + \frac{3}{(s+1)}$$

$$\mathcal{L}^{-1}\left\{ \frac{4s^2-11s+12}{(s-2)^2(s+1)} \right\} = e^{2t} + 2te^{2t} + 3e^{-t}.$$

Note: In general, a factor $(s-r)^m$ in $Q(s)$ gives rise to terms

$$\frac{A_1}{(s-r)} + \frac{A_2}{(s-r)^2} + \cdots + \frac{A_m}{(s-r)^m}$$

in the partial fraction expansion of $\frac{P(s)}{Q(s)}$.

Case 3 Quadratic Factors

Example Find $\mathcal{L}^{-1}\left\{ \dfrac{(5s-4)(s-5)}{(s^2-4s+12)(s+1)} \right\}$

$$\frac{(5s-4)(s-5)}{(s^2-4s+12)(s+1)} = \frac{As+B}{s^2-4s+12} + \frac{C}{s+1}$$

since $s^2-4s+12$ has discr. $\Delta = b^2-4ac$

$$= 16-4\cdot 13 = 16-52 < 0.$$

So

$$(5s-4)(s-5) = (As+B)(s+1) + C(s^2-4s+13).$$

$s=-1 \Rightarrow \quad (-9)(-6) = C(1+4+13)$

$$54 = 18C, \quad C=3.$$

$$(5s-4)(s-5) - 3(s^2-4s+13) = (As+B)(s+1)$$

$$5s^2-29s+20 -3s^2+12s-39 = (As+B)(s+1)$$

$$2s^2-17s-19 = (As+B)(s+1) = As^2+\dots$$

$A = 2$ (coeff of s^2).

$s=0 \Rightarrow \quad -19 = B.$

Hence

$$\frac{(5s-4)(s-5)}{(s^2-4s+13)(s+1)} = \frac{2s-19}{s^2-4s+13} + \frac{3}{s+1}$$

$$s^2-4s+13 = (s-2)^2+9 = (s-2)^2+3^2.$$

$$\frac{2s-19}{s^2-4s+13} = \frac{2s-19}{(s-2)^2+3^2} = \frac{2(s-2)-15}{(s-2)^2+3^2}$$

$$= \frac{2(s-2)}{(s-2)^2+3^2} - \frac{15}{(s-2)^2+3^2}$$

$$= 2\,\frac{(s-2)}{(s-2)^2+3^2} - 5\,\frac{3}{(s-2)^2+3^2}.$$

So

$$\mathcal{L}^{-1}\left\{\frac{2s-19}{s^2-4s+13}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+3^2}\right\} - 5\,\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\}$$

$$= 2\,e^{2t}\cos 3t - 5\,e^{2t}\sin 3t.$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} = 3e^{-t}.$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{(5s-4)(s-5)}{(s^2-4s+13)(s+1)}\right\} = e^{2t}(2\cos 3t - 5\sin 3t) + 3e^{-t}.$$

Note Each irreducible factor $(s^2+bs+c)^m$ gives rise to terms

$$\frac{A_1 s + B_1}{(s^2+bs+c)} + \frac{A_2 s + B_2}{(s^2+bs+c)^2} + \cdots + \frac{A_m s + B_m}{(s^2+bs+c)^m}.$$