7.5 Solving Initial Value Problems

Suggested HW:

Given an IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \; y'(0) = y_1.$$

We let $Y(s) = \mathcal{L}\{y(t)\}$.

Then $\mathcal{L}\{y'(t)\} = sY - y(0) = sY - y_0$

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - sy_0 - y_1.$$

Example (#4) Solve the IVP

$$y'' + 6y' + 5y = 12e^t, \quad y(0) = -1, \; y'(0) = 7.$$

Let $Y = \mathcal{L}\{y\}$. Then

$$\mathcal{L}\{y'\} = sY - y(0) = sY + 1$$

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y + s - 7.$$

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\{12e^t\}$$

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \frac{12}{s-1}$$

$$s^2Y + s - 7 + 6(sY + 1) + 5Y = \frac{12}{s-1}$$

$$(s^2 + 6s + 5)Y + s - 1 = \frac{12}{s-1}$$

$$(s^2 + 6s + 5)Y = \frac{12}{s-1} - (s-1)$$

$$(s+1)(s+5)Y = \frac{12 - (s-1)^2}{(s-1)}$$

$$Y = \frac{12 - (s-1)^2}{(s+1)(s+5)(s-1)}$$

$$\frac{12-(s-1)^2}{(s+1)(s+5)(s-1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+5}$$

$$12-(s-1)^2 = A(s+1)(s+5) + B(s-1)(s+5) + C(s-1)(s+1)$$

$s=1 \Rightarrow \quad 12 = 12A, \quad A=1.$

$s=-1 \Rightarrow \quad 8 = -8B, \quad B=-1.$

$s=-5 \Rightarrow \quad 12-36 = C(-6)(-4)$

$\quad -24 = 24C, \quad C=-1.$

So

$$Y = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5}.$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= e^{t} - e^{-t} - e^{-5t}.$$

<u>Example</u> Solve the IVP

$$y'' + 6y = t^2 - 1, \quad y(0) = 0, \quad y'(0) = -1$$

for $Y = \mathcal{L}\{y\}$.

Let $Y = \mathcal{L}\{y\}$.

$$\mathcal{L}\{y''\} = s^2 Y - s y(0) - y'(0) = s^2 Y + 1$$

$$\mathcal{L}\{t^2-1\} = \frac{2}{s^3} - \frac{1}{s}.$$

$$\mathcal{L}\{y'' + 6y\} = \mathcal{L}\{t^2-1\}$$

$$s^2 Y + 1 + 6Y = \frac{2}{s^3} - \frac{1}{s}$$

$$Y(s^2+6) = \frac{2}{s^3} - \frac{1}{s} - 1$$

$$Y(s) = \frac{1}{s^2+6}\left[\frac{2}{s^3} - \frac{1}{s} - 1\right], \quad \text{for } s>0.$$

#36 $\quad ty'' - ty' + y = 2, \quad y(0)=2, \; y'(0)=-1.$

Let $Y = \mathcal{L}\{y\}$.

$$\mathcal{L}\{y'\} = sY - y(0) = sY - 2.$$

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - 2s + 1$$

$$\mathcal{L}\{ty'\} = -\frac{d}{ds}\mathcal{L}\{y'\} = -\frac{d}{ds}(sY-2)$$

$$= -(sY' + Y)$$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}\mathcal{L}\{y''\} = -\frac{d}{ds}(s^2Y - 2s + 1)$$

$$= -(s^2Y' + 2sY - 2)$$

$$\mathcal{L}\{ty'' - ty' + y\} = \mathcal{L}\{2\}$$

$$-(s^2Y' + 2sY - 2) + (sY' + Y) + Y = \frac{2}{s}$$

$$(-s^2+s)Y' + (2-2s)Y = -2 + \frac{2}{s}$$

$$Y' + \frac{2(1-s)Y}{s(-s+1)} = \frac{-2s+2}{s(s-s^2)}$$

$$Y' + \frac{2}{s}Y = \frac{2(1-s)}{s^2(1-s)} = \frac{2}{s^2}.$$

(21)

We solve this first order linear DE in Y.

An integrating factor is

$$\mu(s) = e^{\int 2/s\, ds} = e^{2\ln s} = s^2.$$

$$\frac{d}{ds}(s^2 Y) = 2sY + s^2 Y'$$
$$= s^2\left(Y' + \frac{2}{s}Y\right)$$
$$= s^2\left(\frac{2}{s^2}\right) = 2.$$

So

$$\frac{d}{ds}(s^2 Y) = 2$$

$$s^2 Y = 2s + c$$
$$Y = \frac{2}{s} + \frac{c}{s^2}$$

$$y = \mathcal{L}^{-1}\{Y\} = 2 + ct$$
$$y(0) = 2.$$
$$y'(t) = c, \quad y'(0) = -1 \text{ so } c = -1 \;\&$$
$$\boxed{y = 2 - t}.$$

Example: (# 12, p.409)

Solve the IVP

$$(*) \quad w'' - 2w' + w = 6t-2, \qquad w(-1)=3, \quad w'(-1)=7$$

using the method of Laplace Transforms.

We define $\quad y(t) = w(t-1)$ so that

$$y'(t) = w'(t-1),$$
$$y''(t) = w''(t-1),$$

and $\quad y(0) = w(-1)$ & $y'(0) = w'(-1)$.

$(*)$

$$\Leftrightarrow \quad w''(t) - 2w'(t) + w(t) = 6t-2, \qquad w(-1)=3, \; w'(-1)=7$$

$$\Leftrightarrow \quad w''(t-1) - 2w'(t-1) + w(t-1) = 6(t-1)-2, \quad w(-1)=3, \; w'(-1)=7$$

$$\Leftrightarrow$$

$$(**) \quad y''(t) - 2y'(t) + y(t) = 6t-8, \qquad y(0)=3, \; y'(0)=7.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$. We take the Laplace transform of both sides of $(**)$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{6t-8\}$$

$$s^2 Y - sy(0) - y'(0) - 2(sY - y(0)) + Y = \frac{6}{s^2} - \frac{8}{s}$$

$$s^2 Y - 3s - 7 - 2(sY - 3) + Y = \frac{6}{s^2} - \frac{8}{s}$$

$$(s^2 - 2s + 1)Y - 3s - 1 = \frac{6}{s^2} - \frac{8}{s}$$

$$Y = \frac{1}{(s-1)^2}\left(3s+1 + \frac{6}{s^2} - \frac{8}{s}\right)$$

$$= \frac{3s+1}{(s-1)^2} + \frac{6}{s^2(s-1)^2} - \frac{8}{s(s-1)^2}$$

2B

$$\frac{3s+1}{(s-1)^2} + \frac{6}{s^2(s-1)^2} - \frac{8}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

(partial fraction)

We multiply both sides by $s^2(s-1)^2$:

$$(3s+1)s^2 + 6 - 8s = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2.$$

$s=0$: $\quad 6 = B$

$s=1$ $\quad 4+6-8 = D$

$$D = 2$$

Coeff of s^3: $\quad 3 = A + C$

Coeff of s^2: $\quad 1 = -2A + B - C + D$

$$= -2A + 6 - C + 2$$

$$A + C = 3$$

$$-2A - C = -7$$

$$-A = -4$$

$$A = 4$$

$$C = -1.$$

So

$$Y = \frac{4}{s} + \frac{6}{s^2} - \frac{1}{(s-1)} + \frac{2}{(s-1)^2}$$

$$y(t) = 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 2\mathcal{L}\left\{\frac{1}{(s-1)^2}\right\}$$

$$y(t) = 4 + 6t - e^t + 2te^t$$

$$w(t) = y(t+1) = 4 + 6(t+1) - e^{t+1} + 2(t+1)e^{t+1}$$

$$\boxed{w(t) = 6t + 10 + e^{t+1} + 2te^{t+1}}$$