Example: (# 12, p.409)

Solve the IVP

$$(*) \quad w'' - 2w' + w = 6t-2, \qquad w(-1)=3, \quad w'(-1)=7$$

using the method of Laplace Transforms.

We define $\quad y(t) = w(t-1)$ so that

$$y'(t) = w'(t-1),$$

$$y''(t) = w''(t-1),$$

and $\quad y(0) = w(-1)$ & $y'(0) = w'(-1)$.

$(*)$

$$\Leftrightarrow \quad w''(t) - 2w'(t) + w(t) = 6t-2, \qquad w(-1)=3, \ w'(-1)=7$$

$$\Leftrightarrow \quad w''(t-1) - 2w'(t-1) + w(t-1) = 6(t-1)-2, \quad w(-1)=3, \ w'(-1)=7$$

$\Leftrightarrow$

$$(**) \quad y''(t) - 2y'(t) + y(t) = 6t-8, \qquad y(0)=3, \ y'(0)=7.$$

Let $\ Y(s) = \mathcal{L}\{y(t)\}$. We take the Laplace transform of both sides of $(**)$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{6t-8\}$$

$$s^2 Y - sy(0) - y'(0) - 2(sY - y(0)) + Y = \frac{6}{s^2} - \frac{8}{s}$$

$$s^2 Y - 3s - 7 - 2(sY - 3) + Y = \frac{6}{s^2} - \frac{8}{s}$$

$$(s^2 - 2s + 1)Y - 3s - 1 = \frac{6}{s^2} - \frac{8}{s}$$

$$Y = \frac{1}{(s-1)^2}\left(3s+1 + \frac{6}{s^2} - \frac{8}{s}\right)$$

$$= \frac{3s+1}{(s-1)^2} + \frac{6}{s^2(s-1)^2} - \frac{8}{s(s-1)^2}$$

$$\frac{3s+1}{(s-1)^2} + \frac{6}{s^2(s-1)^2} - \frac{8}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

(partial fraction)

We multiply both sides by $s^2(s-1)^2$:

$$(3s+1)s^2 + 6 - 8s = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2.$$

$s=0$: $\quad 6 = B$

$s=1$ $\quad 4+6-8 = D$

$D = 2$

Coeff of s^3: $\quad 3 = A + C$

Coeff of s^2: $\quad 1 = -2A + B - C + D$

$= -2A + 6 - C + 2$

$$A + C = 3$$
$$-2A - C = -7$$
$$-A = -4$$
$$A = 4$$
$$C = -1.$$

So

$$Y = \frac{4}{s} + \frac{6}{s^2} - \frac{1}{(s-1)} + \frac{2}{(s-1)^2}$$

$$y(t) = 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 2\mathcal{L}\left\{\frac{1}{(s-1)^2}\right\}$$

$$y(t) = 4 + 6t - e^t + 2te^t$$

$$w(t) = y(t+1) = 4 + 6(t+1) - e^{t+1} + 2(t+1)e^{t+1}$$

$$\boxed{w(t) = 6t + 10 + e^{t+1} + 2te^{t+1}}$$