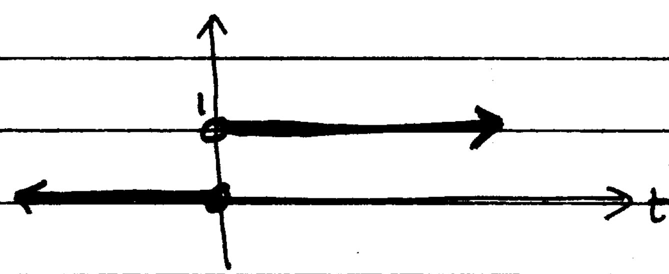
7.6 Transformations of discontinuous & periodic Functions

Suggested HW: odds 1 – 39

The **Heaviside** function is defined by

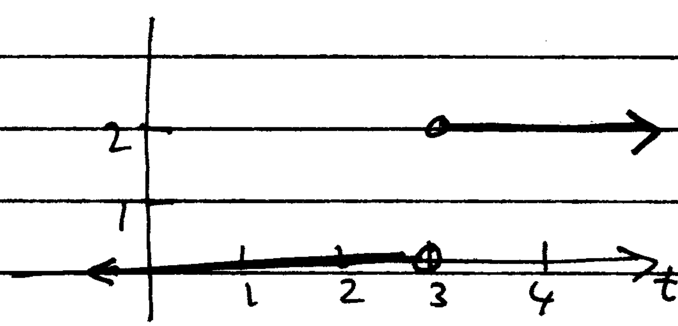
$$H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t>0. \end{cases}$$

$H(t) = u(t)$ is called (also) the unit step function.

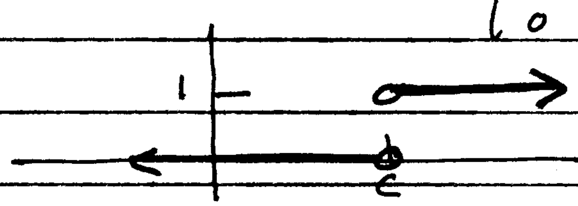


Example Plot $y = 2H(t-3) = \begin{cases} 2 & \text{if } t-3>0 \\ 0 & \text{if } t-3<0 \end{cases}$

$$= \begin{cases} 2 & \text{if } t>3 \\ 0 & \text{if } t<3 \end{cases}$$



In general, $H(t-c) = \begin{cases} 1 & \text{if } t>c \\ 0 & \text{if } t<c \end{cases}$



Let $a \geq 0$.

$$\mathcal{L}\{H(t-a)\} = \mathcal{L}\{u(t-a)\}$$

$$= \int_0^\infty e^{-st} H(t-a)\, dt$$

$$= \int_0^a e^{-st} H(t-a)\, dt + \int_a^\infty e^{-st} H(t-a)\, dt$$

$$= \int_a^\infty e^{-st}\, dt$$

$$\int_a^N e^{-st}\, dt = \frac{e^{-st}}{-s}\Big]_a^N = \frac{e^{-sa}}{s} - \frac{e^{-sN}}{s}$$

$$\lim_{N\to\infty} \int_a^N e^{-st}\, dt = \lim_{N\to\infty} \left(\frac{e^{-sa}}{s} - \frac{e^{-sN}}{s} \right)$$

$$= \frac{e^{-sa}}{s} \qquad \text{if } s > 0.$$

$$\boxed{\mathcal{L}\{H(t-a)\} = \mathcal{L}\{u(t-a)\} = \frac{e^{-sa}}{s} \quad \text{for } s > 0}$$

<u>Theorem</u> Let $a > 0$. Suppose

$$F(s) = \mathcal{L}\{f\} \quad \text{for } s > \alpha \geq 0$$

Then

$$\mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as} F(s).$$

<u>Proof</u>

$$\mathcal{L}\{f(t-a)\, u(t-a)\} = \int_0^\infty f(t-a)\, u(t-a)\, e^{-st}\, dt$$

$$= \int_a^\infty f(t-a)\, e^{-st}\, dt \qquad \left(\text{Let } v = t-a,\ dv = dt\right)$$

$$= \int_0^\infty f(v)\, e^{-s(v+a)}\, dv$$

$$= \int_0^\infty f(v)\, e^{-sv} e^{-sa}\, dv$$

$$= e^{-sa} \int_0^\infty e^{-sv} f(v)\, dv = e^{-sa} F(s). \ \square$$

<u>Corollary.</u> Let $h(t) = g(t+a)$. Then

$$\mathcal{L}\{h(t-a)\, u(t-a)\} = \mathcal{L}\{g(t)\, u(t-a)\}$$

$$= e^{-as}\, \mathcal{L}\{h(t)\}$$

$$= e^{-as}\, \mathcal{L}\{g(t+a)\}$$

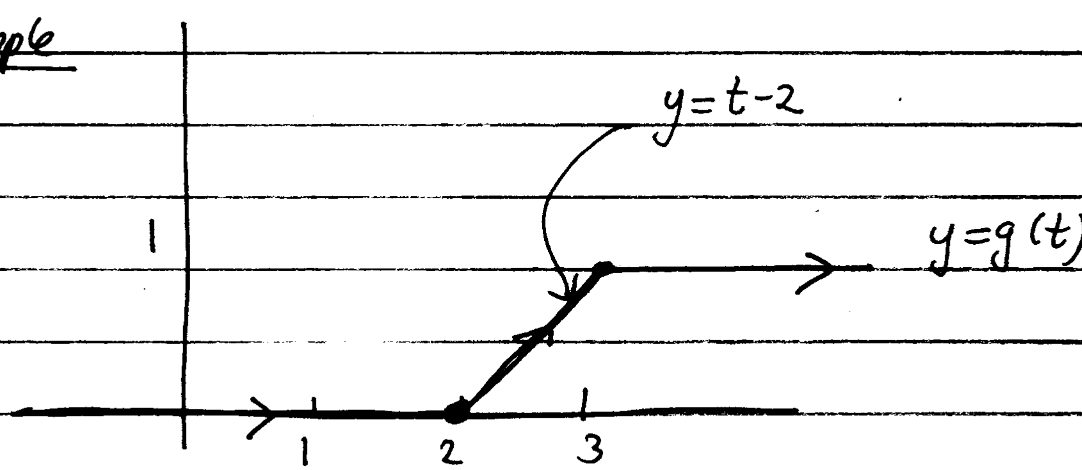
$$\boxed{\mathcal{L}\{g(t)\, u(t-a)\} = e^{-as}\, \mathcal{L}\{g(t+a)\}}$$

Let $0 \leq a < b$.

$u(t-a) = H(t-a)$	0	1	1
$u(t-b) = H(t-b)$	0	0	1

$$u(t-a) - u(t-b) = H(t-a) - H(t-b) = \begin{cases} 1 & \text{if } a < t < b \\ 0 & \text{otherwise} \end{cases}$$

<u>Example</u>



Express $g(t)$ in terms of unit step functions & find $\mathcal{L}\{g(t)\}$.

t	0 to 1	1 to 2	2 to 3	after 3
$u(t-3)$	0	0	0	1
$u(t-2)-u(t-3)$	0	0	1	0
$(t-2)(u(t-2)-u(t-3))$	0	0	$t-2$	0

$$g(t) = (t-2)\,(u(t-2)-u(t-3)) + u(t-3)$$
$$= (t-2)\,u(t-2) - (t-3)\,u(t-3)$$
$$\mathcal{L}\{g(t)\} = \mathcal{L}\{(t-2)u(t-2)\} - \mathcal{L}\{(t-3)\,u(t-3)\}$$
$$= e^{-2s}\mathcal{L}\{t\} - e^{-3s}\mathcal{L}\{t\}$$
$$= \frac{1}{s^2}\left(e^{-2s} - e^{-3s}\right) \quad (\text{for } s>0).$$

Example (#14) Determine $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2+9}\right\}$.

$$\mathcal{L}\{\tfrac{1}{3}\sin 3t\} = \frac{1}{s^2+9}.$$

$\cancel{\mathcal{L}\{u(t-3)\tfrac{1}{3}\sin 3t\} = e^{-3s}}$

$$\mathcal{L}\{u(t-3)\,\tfrac{1}{3}\sin 3(t-3)\} = e^{-3s}\mathcal{L}\{\tfrac{1}{3}\sin 3t\} = \frac{e^{-3s}}{s^2+9}.$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+9}\right\} = \frac{1}{3}u(t-3)\sin 3(t-3).$$

Example (# 18) Determine $\mathcal{L}^{-1}\left\{\frac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}\right\}$

$$\frac{3s^2-s+2}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \quad \text{(partial fracs)}$$

$$3s^2-s+2 = A(s^2+1) + (Bs+C)(s-1).$$

$s=1 \Rightarrow \quad 4 = 2A, \; A=2.$

Coeff of s^2: $\quad 3 = A+B = 2+B, \; B=1.$

$s=0 \Rightarrow \quad 2 = A-C = 2-C, \; C=0.$

So

$$\frac{3s^2-s+2}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{3s^2-s+2}{(s-1)(s^2+1)}\right\} = 2e^t + \cos t = f(t).$$

$$\mathcal{L}\{u(t-1)f(t-1)\} = e^{-s}\mathcal{L}\{f(t)\} = e^{-s}\frac{(3s^2-s+2)}{(s-1)(s^2+1)}.$$

Hence, $\mathcal{L}^{-1}\left\{\frac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}\right\} = u(t-1)f(t-1)$

$$= u(t-1)\left(2e^{t-1} + \cos(t-1)\right).$$

#30 $y'' + y = u(t-2) - u(t-4)$, $y(0)=1, y'(0)=0$

note:

	0		2		4	
$u(t-2)$		0		1		1
$u(t-4)$		0		0		1
$u(t-2)-u(t-4)$		0		1		0

$$g(t) = u(t-2) - u(t-4) = \begin{cases} 1 & \text{if } 2<t<4 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = \mathcal{L}\{y\}$

$$\mathcal{L}\{y''\} = s^2 Y - s\,y(0) - y'(0) = s^2 Y - s$$

$$\mathcal{L}\{u(t-2) - u(t-4)\} = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

$$s^2 Y - s + Y = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

$$(s^2+1)Y = s + \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

$$Y = \frac{s}{s^2+1} + \frac{1}{(s^2+1)s}\left(e^{-2s} - e^{-4s}\right)$$

$$\frac{1}{(s^2+1)(s)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$s=0$ $\quad A = 1$

Coeff of s^2: $\quad 0 = A + B$, $\quad B = -1$

Coeff of s: $\quad 0 = C$.

$$\frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{-s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t = f(t)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s(s^2+1)}$$

$$\mathcal{L}\{u(t-2)f(t-2)\} = e^{-2s}\mathcal{L}\{f\} = \frac{e^{-2s}}{s(s^2+1)}$$

$$\mathcal{L}\{u(t-4)f(t-4) = e^{-4s}\mathcal{L}\{f\} = \frac{e^{-4s}}{s(s^2+1)}.$$

Hence

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s(s^2+1)}\right\}$$

$$= \cos t + \left(1 - \cos(t-2)\right)u(t-2) - \left(1 - \cos(t-4)\right)u(t-4)$$

$$= \begin{cases} \cos t & 0 \le t < 2 \\ 1 + \cos t - \cos(t-2) & 2 < t < 4 \\ \cos t + \cos(t-4) - \cos(t-2) & t > 4 \end{cases}$$

PLOTTING SOLUTION USING MAPLE

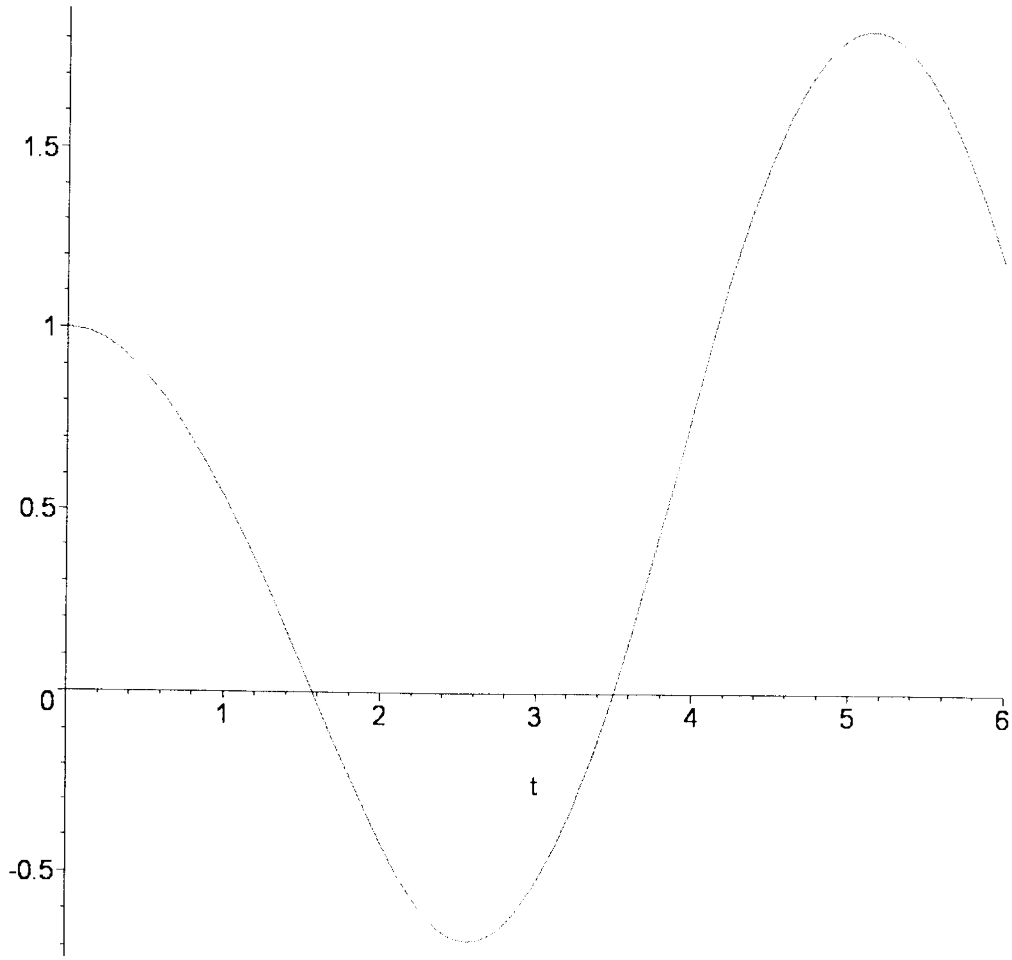

```
> dsolve({diff(y(t),t,t)+y(t)=Heaviside(t-2)-Heaviside(t-4),D(y)(0)=
  0,y(0)=1},y(t));
```

$$y(t) = \cos(t) + (1 - \cos(t-2))\,\text{Heaviside}(t-2) + \text{Heaviside}(t-4)\,(-1 + \cos(t-4))$$

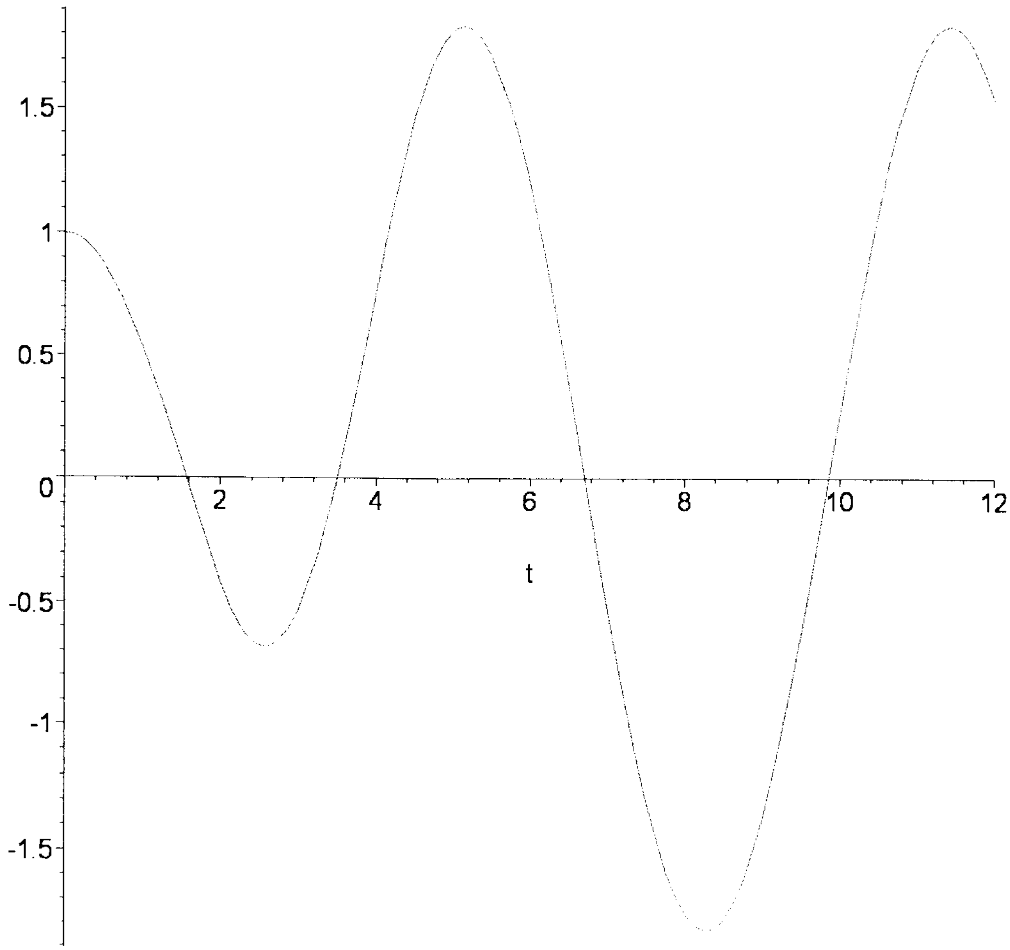
```
> Y:=rhs(%);
```

$$Y := \cos(t) + (1 - \cos(t-2))\,\text{Heaviside}(t-2) + \text{Heaviside}(t-4)\,(-1 + \cos(t-4))$$

```
> plot(Y,t=0..6);
```



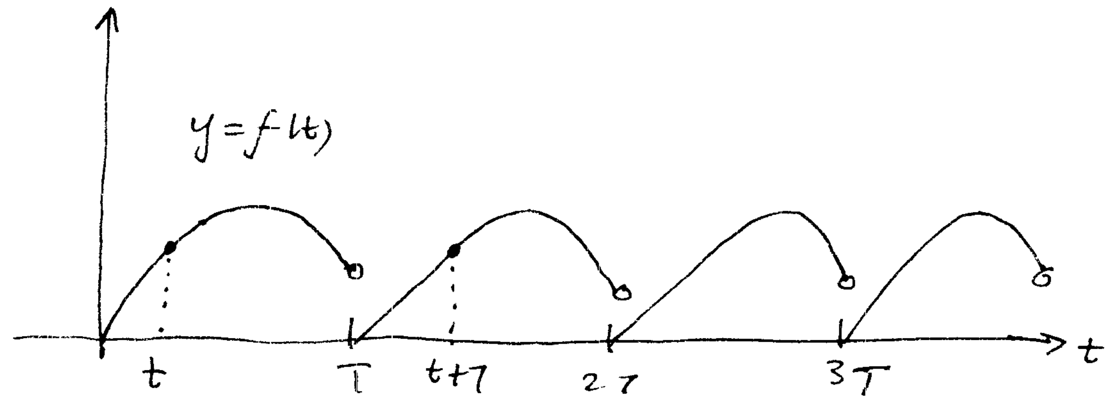

```
> plot(Y,t=0..12);
```


1.5
1
0.5
0
-0.5
-1
-1.5
2
4
6
8
10
12
t

>

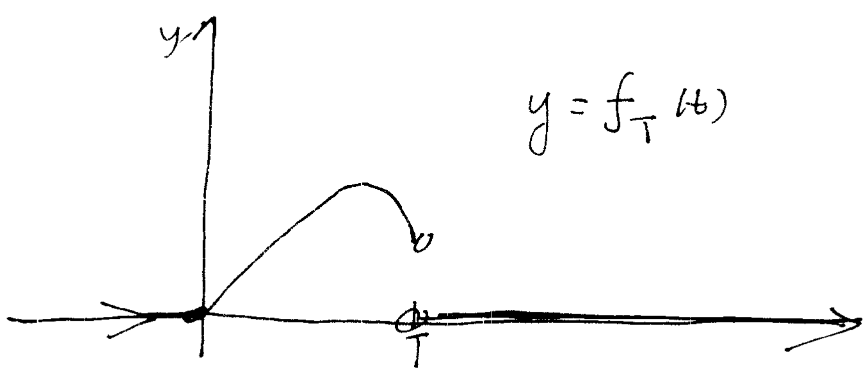
Periodic Functions

Let $T \neq 0$. $f(t)$ has period T if $f(t+T) = f(t)$ for all t.



Let $f_T(t) = f(t)\,(u(t) - u(t-T))$

$$= \begin{cases} f(t) & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$



<u>Theorem</u> Suppose f has period T & is piecewise continuous on $[0,T]$. Then

$$\mathcal{L}\{f\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

Proof.

$$\mathcal{L}\{f_T(t)\} = \mathcal{L}\{f(t)u(t)\} - \mathcal{L}\{\cancel{f}f(t)\,u(t-T)\}$$

$$\int_0^\infty e^{-st} f_T(t)\,dt = \mathcal{L}\{f(t)\} - \mathcal{L}\{f(t)\,u(t-T)\}$$

$$= \mathcal{L}\{f(t)\} - \mathcal{L}\{f(t-T)\,u(t-T)\}$$

since $f(t-T) = f(t-T+T) = f(t)$ for all t.

$$\int_0^\infty e^{-st} f_T\cancel{f}(t)\,dt = \int_0^T e^{-st} f_T(t)\,dt + \int_T^\infty e^{-st} f_T(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt.$$

$$\mathcal{L}\{f(t-T)\,u(t-T)\} = e^{-sT}\mathcal{L}\{f\}.$$

Hence

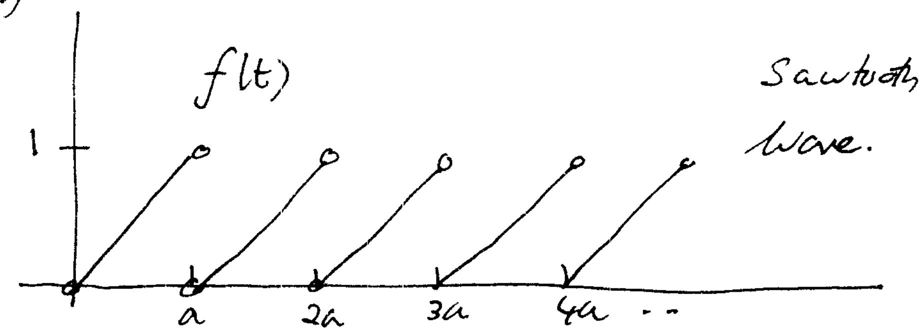
$$\int_0^T e^{-st} f(t)\,dt = \mathcal{L}\{f\} - e^{-sT}\mathcal{L}\{f\}$$

$$= \mathcal{L}\{f\}(1-e^{-sT}),$$

&

$$\mathcal{L}\{f\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

Example :
(#26)



$$f(t) = \frac{t}{a} \qquad 0 < t < a$$

has period $T = a$.

$$\int_0^T e^{-st} f(t)\,dt = \int_0^T e^{-st} \frac{t}{a}\,dt$$

$$= \frac{1}{a} \int_0^T \underbrace{t}_{u} \underbrace{e^{-st}\,dt}_{dv}$$

$$= \frac{1}{a}\left(t\,\frac{e^{-st}}{-s}\Big]_0^T - \int_0^T \frac{e^{-st}}{-s}\,dt \right)$$

$$= \frac{1}{a}\left(\left(-\frac{Te^{-sT}}{s}\right) + \int_0^T \frac{e^{-st}}{s}\,dt \right)$$

$$= \frac{1}{a}\left(-\frac{Te^{-sT}}{s} + \frac{e^{-st}}{-s^2} \right)\Big|_0^T$$

$$= \frac{1}{a}\left(-\frac{Te^{-sT}}{s} + \frac{e^{-sT}}{-s^2} + \frac{1}{s^2} \right)$$

$$= \frac{1}{a}\left(\frac{1}{s^2} - \frac{e^{-sT}}{s^2} - \frac{Te^{-sT}}{s} \right)$$

$$= \frac{1}{a}\left(\frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{a}{s} e^{-sa} \right) \qquad \text{since } (T = a)$$

$$\mathcal{L}\{f\} = \frac{1}{(1-e^{-sa})} \int_0^a e^{-st}\,\frac{t}{a}\,dt$$

$$= \frac{1}{a(1-e^{-sa})s^2}\left(1 - e^{-sa} - as\,e^{-sa}\right)$$