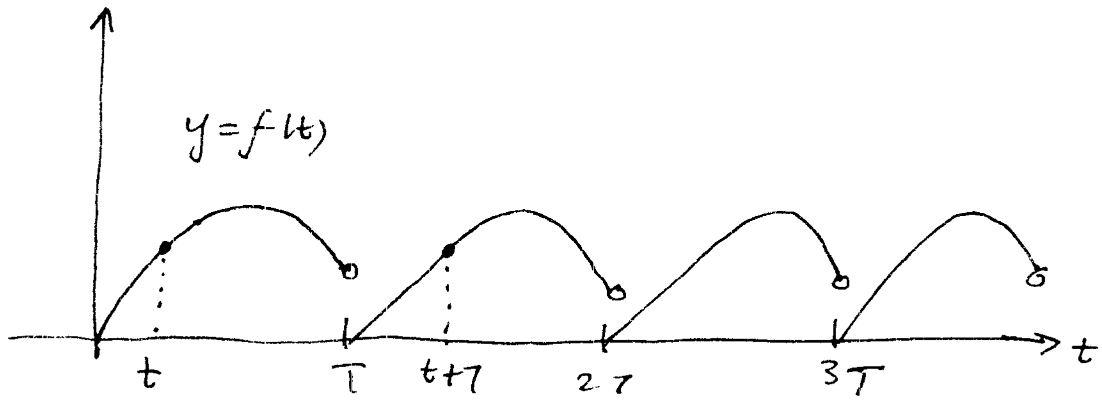
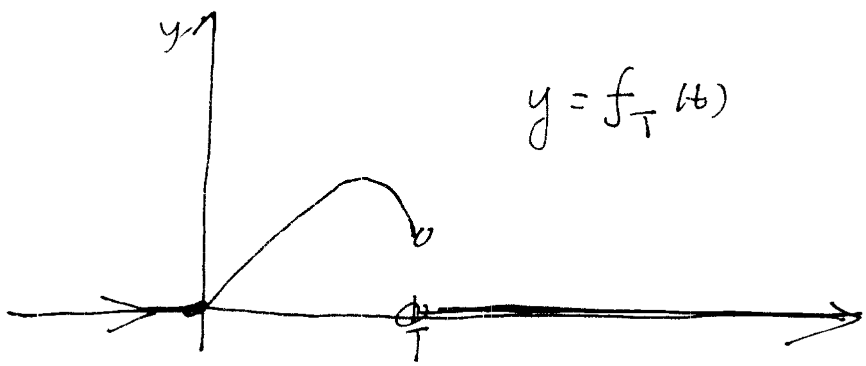
Periodic Functions

Let $T \neq 0$. $f(t)$ has period T if $f(t+T) = f(t)$ for all t.



Let $f_T(t) = f(t)\,(u(t) - u(t-T))$

$$= \begin{cases} f(t) & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$



Theorem Suppose f has period T & is piecewise continuous on $[0,T]$. Then

$$\mathcal{L}\{f\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

Proof.

$$\mathcal{L}\{f_T(t)\} = \mathcal{L}\{f(t)\,u(t)\} - \mathcal{L}\{\cancel{f}(t)\,u(t-T)\}$$

$$\int_0^\infty e^{-st} f_T(t)\,dt = \mathcal{L}\{f(t)\} - \mathcal{L}\{f(t)\,u(t-T)\}$$

$$= \mathcal{L}\{f(t)\} - \mathcal{L}\{f(t-T)\,u(t-T)\}$$

since $f(t-T) = f(t-T+T) = f(t)$ for all t.

$$\int_0^\infty e^{-st} f_T \cancel{f}(t)\,dt = \int_0^T e^{-st} f_T(t)\,dt + \int_T^\infty e^{-st} f_T(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt.$$

$$\mathcal{L}\{f(t-T)\,u(t-T)\} = e^{-sT}\mathcal{L}\{f\}.$$

Hence

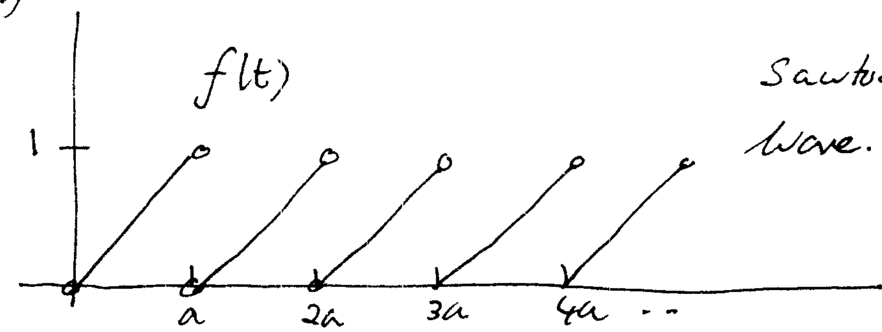
$$\int_0^T e^{-st} f(t)\,dt = \mathcal{L}\{f\} - e^{-sT}\mathcal{L}\{f\}$$

$$= \mathcal{L}\{f\}(1-e^{-sT}),$$

&

$$\mathcal{L}\{f\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

Example :
(#26)



Sawtooth wave.

$$f(t) = \frac{t}{a} \qquad 0 < t < a$$

has period $T = a$.

$$\int_0^T e^{-st} f(t)\,dt = \int_0^T e^{-st} \frac{t}{a}\,dt$$

$$= \frac{1}{a} \int_0^T \underbrace{t}_{u} \underbrace{e^{-st}\,dt}_{dv}$$

$$= \frac{1}{a}\left(t \frac{e^{-st}}{-s} \Big]_0^T - \int_0^T \frac{e^{-st}}{-s}\,dt \right)$$

$$= \frac{1}{a}\left(\left(-\frac{Te^{-sT}}{s} \right) + \int_0^T \frac{e^{-st}}{s}\,dt \right)$$

$$= \frac{1}{a}\left(-\frac{Te^{-sT}}{s} + \frac{e^{-st}}{-s^2} \right)\Big|_0^T$$

$$= \frac{1}{a}\left(-\frac{Te^{-sT}}{s} + \frac{e^{-sT}}{-s^2} + \frac{1}{s^2} \right)$$

$$= \frac{1}{a}\left(\frac{1}{s^2} - \frac{e^{-sT}}{s^2} - \frac{Te^{-sT}}{s} \right)$$

$$= \frac{1}{a}\left(\frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{a\,e^{-sa}}{s} \right) \qquad \text{since } (T = a)$$

$$\mathcal{L}\{f\} = \frac{1}{(1-e^{-sa})} \int_0^a e^{-st} \frac{t}{a}\, dt$$

$$= \frac{1}{a(1-e^{-sa})s^2}\left(1 - e^{-sa} - as\, e^{-sa}\right)$$