Problem: Solve $x'' + 9x = 3\,\delta(t-a)$, $x(0)=1$, $x'(0)=0$.

Let $X = \mathcal{L}\{x\}$.

Then $\mathcal{L}\{x''\} = s^2X - s\,x(0) - x'(0) = s^2X - s$.

$\mathcal{L}\{x'' + 9x\} = \mathcal{L}\{3\,\delta(t-a)\}$.

$$s^2X - s + 9X = 3e^{-as}$$

$$(s^2+9)X = 3e^{-as} + s$$

$$X = \frac{3e^{-as}}{s^2+9} + \frac{s}{s^2+9}$$

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

$$\therefore \mathcal{L}\{u(t-a)\sin 3(t-a)\} = \frac{3e^{-as}}{s^2+9},$$

$$\& \quad \mathcal{L}^{-1}\left\{\frac{3e^{-as}}{s^2+9}\right\} = u(t-a)\sin(3t-3a)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t.$$

Hence, $x(t) = \cos 3t + u(t-a)\sin(3t-3a)$

$$= \begin{cases} \cos 3t & \text{if } t<a \\ \cos 3t + \sin(3t-a) & \text{if } t>a \end{cases}$$

$$= \begin{cases} \cos 3t & \text{if } t<a \\ \cos 3t - \sin 3t & \text{if } t>a. \end{cases}$$