Shifting the Index of Summation

Examples Express the power series as a series with generic term x^k.

(#24)
$$\sum_{n=1}^{\infty} n(n-1)\, a_n x^{n+2}$$

$$= 0 + 2\cdot 1\cdot a_2 x^4 + 3\cdot 2\cdot a_3 x^5 + 4\cdot 3\, a_4 x^6 + \cdots$$

Let $n+2 = k$. Then $n = k-2$

$$\sum_{n=1}^{\infty} n(n-1)\, a_n x^{n+2} \qquad [\text{when } n=1,\ k=3]$$

$$= \sum_{k=3}^{\infty} (k-2)(k-3)\, a_{k-2}\, x^k$$

$$= 0 + 2\cdot 1\cdot a_2 x^4 + 3\cdot 2\cdot a_3 x^5 + \cdots$$

(#26)
$$\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3}$$

$$= \frac{a_1}{4} x^4 + \frac{a_2}{5} x^5 + \frac{a_3}{6} x^6 + \cdots$$

Let $n+3 = k$. The $n = k-3$.

$$\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3} = \sum_{k=4}^{\infty} \frac{a_{k-3}}{k} x^k$$

$$= \frac{a_1}{4} x^4 + \frac{a_2}{5} x^5 + \cdots$$

Let a be an integer. Let m be an integer $m \geq 0$.

$$\sum_{n=m}^{\infty} f(n) = f(m) + f(m+1) + f(m+2) + \cdots$$

Consider change of variable $k = n + a$, $n = k - a$.

$$\sum_{k=m+a}^{\infty} f(k-a) = f(m+a-a) + f(m+a+1-a) + \cdots$$
$$= f(m) + f(m+1) + \cdots$$

#28 Show that

$$2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n\, b_n\, x^{n-1}$$
$$= b_1 + \sum_{n=1}^{\infty} \left(2a_{n-1} + (n+1) b_{n+1}\right) x^n$$

In the first sum we do change index of summation:
$n = k-1 \quad (k \geq 1)$

In the second sum we change index of summation by
$n = k+1 \quad (k \geq 0)$

$$2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n\, b_n\, x^{n-1}$$
$$= 2\sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{k=0}^{\infty} (k+1) b_{k+1} x^k$$
$$= \sum_{k=1}^{\infty} 2a_{k-1} x^k + b_1 + \sum_{k=1}^{\infty} (k+1) b_{k+1} x^k$$
$$= b_1 + \sum_{k=1}^{\infty} 2a_{k-1} x^k + (k+1) b_{k+1} x^k$$

$$= b_1 + \sum_{k=1}^{\infty} (2a_{k-1} + (k+1)b_{k+1})x^k$$

$$= b_1 + \sum_{n=1}^{\infty} (2a_{n-1} + (n+1)b_{n+1})x^n.$$

Check:

$$2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n b_n x^{n-1}$$

$$= 2(a_0 x + a_1 x^2 + a_2 x^3 + \dots)$$

$$+ (b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + \dots)$$

$$= b_1 + (2a_0 + 2b_2)x + (2a_1 + 3b_3)x^2 + (2a_2 + 4b_4)x^3 + \dots$$

$$b_1 + \sum_{n=1}^{\infty} (2a_{n-1} + (n+1)b_{n+1})x^n$$

$$= b_1 + (2a_0 + 2b_2)x^1 + (2a_1 + 3b_3)x^2 + (2a_2 + 4b_4)x^3 + \dots$$

NOTE:

① $\sum_{n=m}^{\infty} f(n) + \sum_{n=m}^{\infty} g(n) = \sum_{n=m}^{\infty} (f(n) + g(n))$

LHS $= (f(m) + f(m+1) + f(m+2) + \dots) + (g(m) + g(m+1) + g(m+2) + \dots)$

RHS $= (f(m) + g(m)) + (f(m+1) + g(m+1)) + (f(m+2) + g(m+2)) + \dots$

② $\sum_{n=m}^{\infty} f(n) = f(m) + \sum_{n=m+1}^{\infty} f(n)$