8.2 Power Series & Analytic Functions

(3) $a_n \searrow 0$, $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

SERIES (1) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv for $p > 1$. (2) $\sum a_n$ conv $\Rightarrow \lim_{n\to\infty} a_n = 0$.

A power series near x_0 (or about x_0) is an infinite series

$$(*) \qquad \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x-x_0) + \cdots$$

where x is a variable & a_n are constants.

$(*)$ converges absolutely if $\sum_{n=0}^{\infty} |a_n (x-x_0)^n|$ converges.

Note If a series converges absolutely then it converges.

Radius of Convergence Theorem

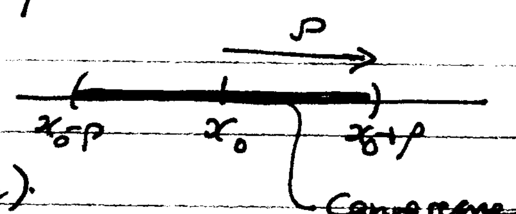
Given a power series $(*)$. There is a real number $\rho \geq 0$

$0 \leq \rho \leq \infty$ (radius of convergence) such that

$(*)$ converges absolutely for $|x - x_0| < \rho$ & diverges for $|x - x_0| > \rho$

$\rho = 0$ Only converges for $x = x_0$.

$\rho = \infty$ (converges absolutely for all x).



Example The series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

converges abs. for $|x| < 1$ & diverges for $|x| > 1$.

In fact $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $-1 < x < 1$.

The series diverges for $|x| \geq 1$.

RATIO TEST

Let $\quad r = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right|.$

If $r < 1$ Then $\sum_{n=1}^{\infty} b_n$ converges absolutely.

If $r > 1$ Then $\sum_{n=1}^{\infty} b_n$ diverges.

RATIO TEST

Let $\quad L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$

Then the radius of convergence of the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is

(1) $\rho = 1/L \quad$ if $L > 0$

(2) $\rho = \infty \quad$ if $L = 0$

(3) $\rho = 0 \quad$ if $L = \infty$

Example (#1) Determine the convergence of $\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} (x-1)^n$.

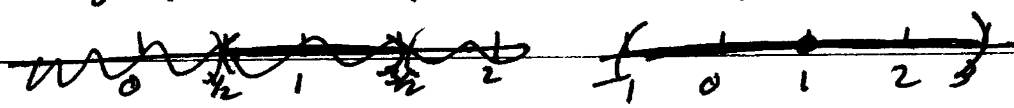
Let $\quad a_n = \frac{2^{-n}}{n+1}.$

$$\frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)}}{n+2} \cdot \frac{n+1}{2^{-n}} = \left(\frac{n+1}{n+2}\right)\left(\frac{1}{2}\right)$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{2} \left(\frac{n+1}{n+2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2} \left(\frac{1 + 1/n}{1 + 2/n}\right) = \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2}.$$

So series converges for $|x-1| < 2$ & div. for $|x-1| > 2$



$x = 3 \quad \sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} (x-1)^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \quad$ diverges

$x = -1 \quad \sum_{n=0}^{\infty} (-1)^n / (n+1) \quad$ converges by Alt. Series test since $1/(n+1) \searrow 0$.

Hence convergence set = $[-1, 3)$.

<u>Theorem</u> Suppose
$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$ has a positive radius of convergence ρ.
Then

(1) f is d'ble for $|x-x_0|<\rho$ &
$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \quad \text{for } |x-x_0|<\rho$$
& series cvges for $|x-x_0|<\rho$.

(2)
$$\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1} + C \text{ for } |x-x_0|<\rho.$$

<u>Cor 1.</u> Suppose $f(x) = \sum a_n (x-x_0)^n$ ~~seems for all~~ has +ve radius of convergence ρ. Then
$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

<u>Cor 2</u> Suppose $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$ for all x in an open interval. Then $a_n = 0$ for all $n \geq 0$.

<u>Example</u> (#12) Find first three nonzero terms in power series expansion of $(\sin x)(\cos x)$.
$$\sin x = x - \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sin x \cos x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$= x + \left(-\tfrac{1}{2} - \tfrac{1}{6}\right)x^3 + \left(\frac{1}{4!} + \frac{1}{3!\,2!} + \frac{1}{5!}\right)x^5 + \cdots$$

$$= x - \tfrac{2}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

Theorem

Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, $g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$ converge for $|x-x_0| < \rho$. Then

$$f(x)\,g(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$ converges for $|x-x_0| < \rho$ &

$$c_n = \sum_{k=0}^{n} a_k \, b_{n-k} \, .$$ (Cauchy products).

Definition A function $f(x)$ is analytic at x_0 if $\cancel{f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n}$ some power series near x_0 with +ve radius of convergence converges to $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{for } |x-x_0| < \rho.$$

Note If f is analytic at $x = x_0$ then

$$a_n = \frac{f^{(n)}(x_0)}{n!} \quad \text{for } n \geq 0.$$

Shifting the Index of Summation

Examples Express the power series as a series with generic term x^k.

(#24) $$\sum_{n=1}^{\infty} n(n-1)\, a_n x^{n+2}$$

$$= 0 + 2\cdot 1\cdot a_2 x^4 + 3\cdot 2\cdot a_3 x^5 + 4\cdot 3\, a_4 x^6 + \cdots$$

Let $n+2 = k$. Then $n = k-2$

$$\sum_{n=1}^{\infty} n(n-1)\, a_n x^{n+2}$$ [when $n=1$, $k=3$]

$$= \sum_{k=3}^{\infty} (k-2)(k-3)\, a_{k-2}\, x^k$$

$$= 0 + 2\cdot 1\cdot a_2 x^4 + 3\cdot 2\cdot a_3 x^5 + \cdots$$

(#26) $$\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3}$$

$$= \frac{a_1}{4} x^4 + \frac{a_2}{5} x^5 + \frac{a_3}{6} x^6 + \cdots$$

Let $n+3 = k$. The $n = k-3$.

$$\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3} = \sum_{k=4}^{\infty} \frac{a_{k-3}}{k} x^k$$

$$= \frac{a_1}{4} x^4 + \frac{a_2}{5} x^5 + \cdots$$

Let a be an integer. Let m be an integer $m \geq 0$.

$$\sum_{n=m}^{\infty} f(n) = f(m) + f(m+1) + f(m+2) + \cdots$$

Consider change of variable $k = n + a$, $n = k - a$.

$$\sum_{k=m+a}^{\infty} f(k-a) = f(m+a-a) + f(m+a+1-a) + \cdots$$
$$= f(m) + f(m+1) + \cdots$$

#28 Show that

$$2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n\, b_n\, x^{n-1}$$
$$= b_1 + \sum_{n=1}^{\infty} \left(2a_{n-1} + (n+1) b_{n+1}\right) x^n$$

In the first sum we do change index of summation: $n = k-1$ $(k \geq 1)$

In the second sum we change index of summation by $n = k+1$ $(k \geq 0)$

$$2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n\, b_n\, x^{n-1}$$
$$= 2\sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{k=0}^{\infty} (k+1) b_{k+1} x^k$$
$$= \sum_{k=1}^{\infty} 2a_{k-1} x^k + b_1 + \sum_{k=1}^{\infty} (k+1) b_{k+1} x^k$$
$$= b_1 + \sum_{k=1}^{\infty} 2a_{k-1} x^k + (k+1) b_{k+1} x^k$$

$$= b_1 + \sum_{k=1}^{\infty} (2a_{k-1} + (k+1)b_{k+1})x^k$$

$$= b_1 + \sum_{n=1}^{\infty} (2a_{n-1} + (n+1)b_{n+1})x^n$$

Check:

$$2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n b_n x^{n-1}$$

$$= 2(a_0 x + a_1 x^2 + a_2 x^3 + \cdots)$$

$$+ (b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + \cdots)$$

$$= b_1 + (2a_0 + 2b_2)x + (2a_1 + 3b_3)x^2 + (2a_2 + 4b_4)x^3 + \cdots$$

$$b_1 + \sum_{n=1}^{\infty} (2a_{n-1} + (n+1)b_{n+1})x^n$$

$$= b_1 + (2a_0 + 2b_2)x^1 + (2a_1 + 3b_3)x^2 + (2a_2 + 4b_4)x^3 + \cdots$$

NOTE :

① $$\sum_{n=m}^{\infty} f(n) + \sum_{n=m}^{\infty} g(n) = \sum_{n=m}^{\infty} (f(n) + g(n))$$

$$LHS = (f(m) + f(m+1) + f(m+2) + \cdots) + (g(m) + g(m+1) + g(m+2) + \cdots)$$

$$RHS = (f(m) + g(m)) + (f(m+1) + g(m+1)) + (f(m+2) + g(m+2)) + \cdots$$

② $$\sum_{n=m}^{\infty} f(n) = f(m) + \sum_{n=m+1}^{\infty} f(n)$$