Review of Power Series

The Geometric Series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

converges for $|x| < 1$ &

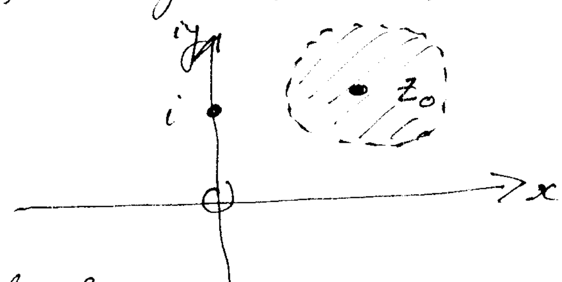
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

<u>Definition</u> Let $f(z)$ be a complex function. $f(z)$ is <u>analytic</u> at $z = z_0$ if it has a power series expansion

$$f(z) = a_0 + a_1 (z - z_0)^1 + a_2 (z - z_0)^2 + a_3 (z - z_0)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for } |z - z_0| < R$$

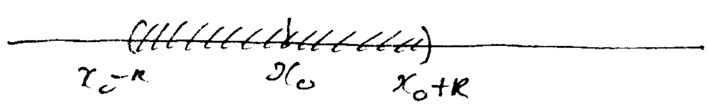
where R is some positive number.



Usually we will restrict to real functions

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $|x - x_0| < R$, some $R > 0$. Hence $a_n \in \mathbb{R}$ for $n \geq 0$.



<u>Theorem</u> Let $R>0$

(1) If $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for $|x-x_0|<R$, then

(a) $$a_0 = f(x_0), \quad a_n = \frac{f^{(n)}(x_0)}{n!} \quad \text{for } n \geq 1.$$

(b) $$f'(x) = \sum_{n=0}^{\infty} a_n \, n(x-x_0)^{n-1}$$
$$= \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \quad \text{for } |x-x_0|<R.$$

(c) $$\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1} + C \quad \text{for } |x-x_0|<R.$$

(2) Any rational function
$$R(x) = \frac{p(x)}{q(x)} \quad (p(x), q(x) \text{ polynomials})$$
is analytic at every point x_0 where $q(x_0) \neq 0$.

(3) The functions
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
are ~~any~~ analytic everywhere.

Example

Let $R(x) = \dfrac{x^2 - x - 2}{x^2 + 3x + 2}$. Where is $R(x)$ analytic?

$x^2 + 3x + 2 = (x+1)(x+2) = 0$ when $x = -1, -2$.

So $R(x)$ is analytic everywhere except at $x = -1, -2$.

The points $x = -1, -2$ are called singularities.

$$R(x) = \frac{x^2 - x - 2}{x^2 + 3x + 2}$$

$$= \frac{(x-2)(x+1)}{(x+1)(x+2)}$$

$$= \frac{x-2}{x+2} \qquad (\text{if } x \neq -1, -2).$$

So $x = -1$ is called a removable singularity.

8.3 Power Series Sol^ns to Linear DEs

Consider a linear homogeneous 2^{nd} order DE

$$(*) \quad a_2(x)\, y''(x) + a_1(x) y'(x) + a_0(x)\, y(x) = 0.$$

$\Leftrightarrow$

$$(**) \quad y''(x) + p(x) y'(x) + q(x)\, y(x) = 0$$

where $p(x) = \frac{a_1(x)}{a_2(x)}$, $q(x) = \frac{a_0(x)}{a_2(x)}$ (provided $a_2(x) \neq 0$).

<u>Definition</u> A point x_0 is called an <u>ordinary point</u> of $(*)$ if both $p(x) = \frac{a_1(x)}{a_2(x)}$ & $q(x) = \frac{a_0(x)}{a_2(x)}$ are analytic at x_0. If x_0 is not an ordinary point it is called a <u>singular point</u> of $(*)$.

<u>Example</u> Determine the singular points of

$$(x^2+3x+2)\, y''(x) + (x+1) y'(x) + (x^2-x-2) y(x) = 0$$

$$\Leftrightarrow \quad y''(x) + \frac{(x+1)}{(x^2+3x+2)} y'(x) + \frac{(x^2-x-2)}{(x^2+3x+2)} y(x) = 0.$$

$$p(x) = \frac{(x+1)}{(x^2+3x+2)} = \frac{(x+1)}{(x+1)(x+2)}, \quad q(x) = \frac{(x+1)(x-2)}{(x+1)(x+2)}$$

There are singularities at $x = -1, -2$. However $x = -1$ is a removable singularity.

$$p(x) = \frac{1}{x+2}, \quad q(x) = \frac{x-2}{(x+2)}$$

are analytic everywhere except at $x = -2$.

$x = -2$ is the only <u>singular point</u>.

Example (a) Find at least the first four non-zero terms in a power series expansion about $x=0$ for the general solution of the DE

$$y''(x) - 2x\,y'(x) - 2y(x) = 0.$$

$p(x) = -2x$, $q(x) = -2$ are analytic everywhere.

$\boxed{\text{Note at any ordinary point } (*) \text{ will have two linearly independent analytic solns near } x=0.}$

Let
$$y = \sum_{n=0}^{\infty} a_n x^n, \quad -2y = \sum_{n=0}^{\infty} (-2a_n) x^n$$

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

$$-2x\,y' = \sum_{n=1}^{\infty} -2n\,a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

$$= \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2}\, x^m \qquad \left(\begin{matrix} n-2=m \\ n=m+2 \end{matrix}\right)$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n.$$

Hence
$$y'' - 2xy' - 2y = \sum_{n=0}^{\infty} \Big((n+2)(n+1)\,a_{n+2} - 2n\,a_n - 2a_n\Big) x^n = 0$$

$$\Leftrightarrow \quad (n+2)(n+1)\,a_{n+2} - 2n\,a_n - 2a_n = 0 \text{ for all } n \geq 0$$

$$(n+2)(n+1)\,a_{n+2} - (2n+2)\,a_n = 0$$

$$(n+2)\,a_{n+2} - 2a_n = 0 \text{ for all } n \geq 0$$

$$a_{n+2} = \frac{2a_n}{n+2} \text{ for all } n \geq 0.$$

We want two linearly independent solns.

For $\boxed{y_1}$ we want $y_1(0)=1$, $y_1'(0)=0$.

$$a_0 = 1$$
$$a_1 = 0$$
$$a_2 = \frac{2a_0}{2} = 1$$
$$a_3 = \frac{2a_1}{3} = 0$$
$$a_4 = \frac{2a_2}{4} = 1/2$$

Hence

$$y_1(x) = 1 + x^2 + \tfrac{1}{2}x^4 + \cdots$$

For $\boxed{y_2}$ $y_2(0) = 0$, $y_2'(0) = 1$.

Now
$$a_0 = 0$$
$$a_1 = 1$$
$$a_2 = \frac{2a_0}{2} = 0$$
$$a_3 = \frac{2a_1}{3} = 2/3$$
$$a_4 = \frac{2a_2}{4} = 0$$
$$a_5 = \frac{2a_3}{5} = \frac{2^2}{15} = \frac{4}{15}$$

Hence $y_2(x) = x + 2/3\, x^3 + 4/15\, x^5 + \cdots$

Note $y_1(x), y_2(x)$ are linearly indept. Since
$$W[y_1,y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

The general sol$\underline{n}$ is given by

$$y = c_1\left(1 + x^2 + \tfrac{1}{2}x^4 + \cdots\right)$$
$$\quad + c_2\left(x + \tfrac{2}{3}x^3 + \tfrac{4}{15}x^5 + \cdots\right)$$

for any constants c_1, c_2.

(b) Repeat problem but find a general formula for the coefficients.

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

We found $\quad a_{n+2} = \dfrac{2a_n}{n+2} \quad$ for $n \geq 0$.

~~For y_1, $a_0=1$, $a_1=0$.~~

$$a_2 = a_0$$
$$a_3 = \frac{2}{3}a_1$$
$$a_4 = \frac{2}{4}a_2 = \frac{1}{2}a_0$$
$$a_5 = \frac{2}{5}a_3 = \frac{2}{5}\cdot\frac{2}{3}\cdot a_1$$
$$a_6 = \frac{2}{6}a_4 = \frac{1}{3}\cdot\frac{1}{2}\cdot a_0$$
$$a_7 = \frac{2}{7}a_5 = \frac{2}{7}\cdot\frac{2}{5}\cdot\frac{2}{3}\cdot a_1$$
$$a_8 = \frac{2}{8}a_6 = \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot a_0$$

$$a_{2n} = \frac{1}{n!} \, a_0$$

$$a_{2n+1} = \frac{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}{(2n+1)(2n-1)\cdots\cdots 3} \, a_1 = \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \, a_1$$

Hence

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} a_0 + \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1} a_1$$

$$= a_0 \, y_1 + a_1 \, y_2 \qquad (a_0, a_1 \text{ any constants})$$

&

$$y_1 = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = e^{x^2}$$

$$y_2 = \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}.$$

Example

(a) Find the 1st four nonzero terms of the general soln of

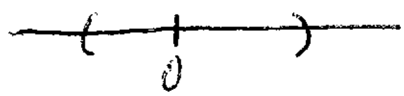
$$y'' - 2xy' - 2y = 0$$

as a power series about $x=0$

$P(x) = -2x$

$Q(x) = -2$ are analytic everywhere.

Since they are analytic at $x=0$ the gen soln is analytic at $x=0$ ie has a power series valid for $|x| < P$ (some P).



Method 1

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7 x^6 + \cdots$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \cdots$

$-2xy' = -2a_1 x - 4a_2 x^2 - 6a_3 x^3 - 8a_4 x^4 - 10a_5 x^5$

$-2y = -2a_0 - 2a_1 x - 2a_2 x^2 - 2a_3 x^3 - 2a_4 x^4 - 2a_5 x^5 + \cdots$

$$= (2a_2 - 2a_0) + (6a_3 - 4a_1)x$$
$$+ (12a_4 - 6a_2)x^2 + (20a_5 - 8a_3)x^3$$
$$+ (30a_6 - 10a_4)x^4 + (42a_7 - 12a_5)x^5 + \cdots$$

So

$2a_2 - 2a_0 = 0$	$a_2 = a_0$
$6a_3 - 4a_1 = 0$	$a_3 = 2/3\, a_1$
$12a_4 - 6a_2 = 0$	$a_4 = 1/2\, a_2$
$20a_5 - 8a_3 = 0$	$a_5 = 2/5\, a_3$
$30a_6 - 10a_4 = 0$	$a_6 = 1/3\, a_4$
$42a_7 - 12a_5 = 0$	$a_7 = 12/42\, a_5$
	$= 2/7\, a_5$

$a_2 = a_0$
$a_3 = 2/3\, a_1$
$a_4 = 1/2\, a_0$
$a_5 = 2/5\,(2/3)\, a_1$
$a_6 = 1/3\,(1/2)\, a_2$
$a_7 = 2/7\,(2/5)\, a_3 = 2/7\,(2/5)(2/3)\, a_1$

$a_3 = 2/3\, a_1$
$a_5 = (2/5)(2/3)\, a_1$
$a_7 = (2/7)(2/5)(2/3)\, a_1$
$a_9 = (2/9)(2/7)(2/5)(2/3)\, a_1$

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$$y = a_0 + a_1 x + a_0 x^2 + 2/3\, a a_1 x^3 + (1/2) a_0 x^4 + (2/3)(2/3) a_1 x^5 + (1/3)(1/2) a_0 x^6 + (2/7)(2/5)(2/3) a_1 x^7 + (1/4)(1/3)(1/2) a_0 x^8 + \cdots$$

$$= a_0 \left(1 + x^2 + 1/2\, x^4 + (1/3)(1/2) x^6 + (1/4)(1/3)(1/2) x^8\right) + (2/7)(2/5)(2/3) a_1 x^7 + \cdots$$

$$+ a_1 \left(x + 2/3\, x^3 + (2/5)(2/3) x^5 + (2/7)(2/5)(2/3) x^7 \right.$$

$$+ (1/4)(1/3)(1/2) a_0 x^8 + \cdots$$

where a_0 a_1 are any constants.

Note (1) $y_1 = 1 + x^2 + 1/2\, x^4 + \cdots$

$y_2 = x + 2/3\, x^3 + (2/5)(2/3) x^5 + \cdots$

are 2 linearly indept soln of the homogeneous de.

(2)

$$W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Method 2

(b) Find a recurrence for the coefficients

(c) Find general formula for the coefficients

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$-2xy' = \sum_{n=0}^{\infty} (-2) n a_n x^n$$

$$-2y = \sum_{n=0}^{\infty} (-2) a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{\boxed{n-2}}$$ We want x^n

Let $k = n-2$ $n = k+2$

$$y'' = \sum (k+2)(k+1) a_{k+2} x^k$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$$

(5)

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+1}x^n$$

$$y'' - 2xy' - 2y = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} -2na_nx^n + \sum_{n=0}^{\infty} -2a_nx^n$$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - 2na_n - 2a_n\right)x^n$$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - 2(n+1)a_n\right)x^n = 0$$

Hence

$(n+2)(n+1)a_{n+2} - 2(n+1)a_n = 0$ for all $n \geq 0$

(b) recurrence

$$(n+2)(n+1)a_{n+2} = \frac{2(n+1)a_n}{(n+2)(n+1)} \quad \left(\text{since } (n+2)(n+1) \neq 0\right)$$

$$a_{n+2} = \frac{2a_n}{n+2} \quad \text{for } n \geq 0$$

(c)

$a_2 = a_0$

$a_4 = \frac{1}{2}a_0$

$a_6 = (\frac{1}{3})(\frac{1}{2})a_0$

$a_8 = (\frac{1}{4})(\frac{1}{3})(\frac{1}{2})a_0$

$$a(2n) = \frac{1}{(n)(n-1)\dots(2)}a_0 = \frac{a_0}{n!} \quad \text{for } n \geq 0$$

$a_3 = (2/3)(a_1)$

$a_5 = (2/5)(2/3) a_1$

$a_7 = (2/7)(2/5)(2/3) a_1$

$a(2n+1) = \dfrac{2}{(2n+1)(2n-1)\ldots(3)} a_1$

n	$2n+1$	2^n
0	1	1
1	3	2^1
2	5	2^2
3	7	2^3
4	9	2^4

d) write general soln ~~as~~ using the general formula for the coefficients

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \ldots) + (a_1 x + a_3 x^3 + a_5 x^5 + \ldots)$$

$$= \sum_{n=0}^{\infty} (a_{2n} x^{2n}) + \sum_{n=0}^{\infty} (a_{2n+1} x^{2n+1})$$

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$$= \sum_{n=0}^{\infty} \frac{a_0}{n!} x^{2n} + \sum_{n=0}^{\infty} \frac{2 a_1 x^{2n+1}}{(2n+1)(2n-1)\ldots(3)(1)}$$

$$y = a_0\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right) + a_1\left(\sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{(2n+1)(2n-1)\ldots(3)}\right)$$

where a_0, a_1 are any constants

note: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{{x^2}^n}{n!}$$

(Bonus) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = e^{x^2}$

(21) $y'' - xy' + 4y = 0$.

$p(x) = -x$, $q(x) = 4$ are analytic everywhere & $x=0$ is an ordinary point.

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$4y = \sum_{n=0}^{\infty} 4 a_n x^n$$

$$-xy' = \sum_{n=0}^{\infty} (-n a_n)\, x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n\, x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n$$

$$y'' - xy' + 4y = \sum_{n=0}^{\infty} \big((n+2)(n+1)\, a_{n+2} - n a_n + 4 a_n\big) x^n = 0$$

So $(n+2)(n+1)\, a_{n+2} + (-n+4) a_n = 0$ for all $n \geq 0$,

& $$a_{n+2} = \frac{(n-4)\, a_n}{(n+1)(n+2)} \text{ for } n \geq 0.$$

$$a_2 = \frac{-4}{2} a_0 = -2a_0$$

$$a_3 = \frac{(-3)\, a_1}{3 \cdot 2}$$

$$a_4 = \frac{-2\, a_2}{4 \cdot 3} = -\frac{1}{6} a_2 = \tfrac{1}{3} a_0$$

$$a_5 = \frac{(-1)\, a_3}{5 \cdot 4} = \frac{(-3)(-1)}{5 \cdot 4 \cdot 3 \cdot 2} a_1$$

$$a_6 = 0$$

$$a_7 = \frac{1 \cdot a_5}{7 \cdot 6} = \frac{(-3)(-1)(1)\, a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$

$$a_8 = 0$$

$$a_9 = \frac{(+3)\, a_7}{9 \cdot 8} = \frac{(-3)(-1)(1)(3)\, a_1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$

We see that

$a_2 = -2a_0$, $a_4 = \tfrac{1}{3} a_0$, and $a_{2n} = 0$ for $n \geq 3$.

Also

$$a_{2n+1} = \frac{(-3)(-1)(1) \cdots (2n-5)}{(2n+1)!} a_1 \quad \text{for } n \geq 0.$$

Therefore

$$y = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= \left(a_0 - 2a_0 x^2 + \tfrac{1}{3} a_0 x^4\right) + \sum_{n=0}^{\infty} \frac{(-3)(-1) \cdots (2n-5)\, a_1 x^{2n+1}}{(2n+1)!}$$

$$= a_0\left(1 - 2x^2 + \tfrac{1}{3} x^4\right) + a_1 \sum_{n=0}^{\infty} \frac{(-3)(-1) \cdots (2n-5)\, x^{2n+1}}{(2n+1)!}$$

where a_0, a_1 are any constants.