8.3 Power Series Solns to Linear DEs

Consider a linear homogeneous 2^{nd} order DE

(*) $\quad a_2(x)\, y''(x) + a_1(x) y'(x) + a_0(x)\, y(x) = 0.$

$\Leftrightarrow$

(**) $\quad y''(x) + p(x) y'(x) + q(x)\, y(x) = 0$

where $p(x) = \dfrac{a_1(x)}{a_2(x)}$, $q(x) = \dfrac{a_0(x)}{a_2(x)}$ (provided $a_2(x) \neq 0$).

<u>Definition</u> A point x_0 is called an <u>ordinary point</u> of (*) if both $p(x) = \dfrac{a_1(x)}{a_2(x)}$ & $q(x) = \dfrac{a_0(x)}{a_2(x)}$ are analytic at x_0. If x_0 is not an ordinary point it is called a <u>singular point</u> of (*).

<u>Example</u> Determine the singular points of

$$(x^2+3x+2)\, y''(x) + (x+1) y'(x) + (x^2-x-2) y(x) = 0$$

$$\Leftrightarrow \quad y''(x) + \frac{(x+1)}{(x^2+3x+2)}\, y'(x) + \frac{(x^2-x-2)}{(x^2+3x+2)}\, y(x) = 0.$$

$$p(x) = \frac{(x+1)}{(x^2+3x+2)} = \frac{(x+1)}{(x+1)(x+2)}, \quad q(x) = \frac{(x+1)(x-2)}{(x+1)(x+2)}$$

There are singularities at $x = -1, -2$. However $x = -1$ is a removable singularity.

$$p(x) = \frac{1}{x+2}, \quad q(x) = \frac{x-2}{(x+2)}$$

are analytic everywhere except at $x = -2$.

$x = -2$ is the only <u>singular point</u>.