Example (a) Find at least the first four non-zero terms in a power series expansion about $x=0$ for the general solution of the DE

$$y''(x) - 2x\,y'(x) - 2y(x) = 0.$$

$p(x) = -2x$, $q(x) = -2$ are analytic everywhere.

$\boxed{\text{Note at any ordinary point } (*) \text{ will have two linearly independent analytic solns near } x=0.}$

Let
$$y = \sum_{n=0}^{\infty} a_n x^n, \quad -2y = \sum_{n=0}^{\infty} (-2a_n) x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$-2x\,y' = \sum_{n=1}^{\infty} -2n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m \qquad \left(\begin{array}{c} n-2=m \\ n=m+2 \end{array}\right)$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Hence
$$y'' - 2xy' - 2y = \sum_{n=0}^{\infty} \Big((n+2)(n+1) a_{n+2} - 2n a_n - 2a_n \Big) x^n = 0$$

$$\leftrightarrow \quad (n+2)(n+1) a_{n+2} - 2n a_n - 2a_n = 0 \text{ for all } n \geq 0$$

$$(n+2)(n+1) a_{n+2} - (2n+2) a_n = 0$$

$$(n+2) a_{n+2} - 2a_n = 0 \text{ for all } n \geq 0$$

$$a_{n+2} = \frac{2a_n}{n+2} \text{ for all } n \geq 0.$$