Example

(a) Find the 1st four nonzero terms of the general soln of

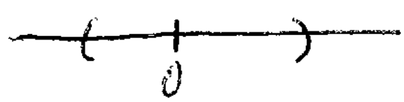
$$y'' - 2xy' - 2y = 0$$

as a power series about $x = 0$

$P(x) = -2x$

$Q(x) = -2$ are analytic everywhere.

Since they are analytic at $x=0$ the gen soln is analytic at $x=0$ ie has a power series valid for $|x| < P$ (some P).



Method 1

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7 x^6 + \ldots$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \ldots$

$-2xy' = -2a_1 x - 4a_2 x^2 - 6a_3 x^3 - 8a_4 x^4 - 10a_5 x^5$

$-2y = -2a_0 - 2a_1 x - 2a_2 x^2 - 2a_3 x^3 - 2a_4 x^4 - 2a_5 x^5 + \ldots$

$$= (2a_2 - 2a_0) + (6a_3 - 4a_1)x$$
$$+ (12a_4 - 6a_2)x^2 + (20a_5 - 8a_3)x^3$$
$$+ (30a_6 - 10a_4)x^4 + (42a_7 - 12a_5)x^5 + \cdots$$

so

$$2a_2 - 2a_0 = 0 \qquad a_2 = a_0$$
$$6a_3 - 4a_1 = 0 \qquad a_3 = 2/3\, a_1$$
$$12a_4 - 6a_2 = 0 \qquad a_4 = 1/2\, a_2$$
$$20a_5 - 8a_3 = 0 \qquad a_5 = 2/5\, a_3$$
$$30a_6 - 10a_4 = 0 \qquad a_6 = 1/3\, a_4$$
$$42a_7 - 12a_5 = 0 \qquad a_7 = 12/42\, a_5$$
$$= 2/7\, a_5$$

$$a_2 = a_0$$
$$a_3 = 2/3\, a_1$$
$$a_4 = 1/2\, a_0$$
$$a_5 = 2/5\,(2/3)\,a_1$$
$$a_6 = 1/3\,(1/2)\,a_2$$
$$a_7 = 2/7\,(2/5)\,a_3 = 2/7\,(2/5)(2/3)\,a_1$$

$$a_3 = 2/3\, a_1$$
$$a_5 = (2/5)(2/3)\,a_1$$
$$a_7 = (2/7)(2/5)(2/3)\,a_1$$
$$a_9 = (2/9)(2/7)(2/5)(2/3)\,a_1$$

$$y = a_0 + a_1 x + a_0 x^2 + 2/3\,a a_1 x^3 + (1/2) a_0 x^4 + (2/3)(2/3) a_1 x^5 + (1/3)(1/2) a_0 x^6 + (2/7)(2/5)(2/3) a_1 x^7 + (1/4)(1/3)(1/2) a_0 x^8 + \cdots$$

$$= a_0 \left(1 + x^2 + 1/2\, x^4 + (1/3)(1/2) x^6 + (1/4)(1/3)(1/2) x^8\right) + (2/7)(2/5)(2/3) a_1 x^7 + \cdots$$

$$+ a_1 \left(x + 2/3\, x^3 + (2/5)(2/3) x^5 + (2/7)(2/5)(2/3) x^7 \right.$$

$$+ (1/4)(1/3)(1/2) a_0 x^8 + \cdots$$

where a_0 a_1 are any constants.

<u>Note</u> (1) $y_1 = 1 + x^2 + 1/2\, x^4 + \cdots$

$y_2 = x + 2/3\, x^3 + (2/5)(2/3) x^5 + \cdots$

are 2 linearly indep't soln of the homogeneous de.

(2)

$$W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Method 2

(b) Find a recurrence for the coefficients

(c) Find general formula for the coefficients

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$-2xy' = \sum_{n=0}^{\infty} (-2) n a_n x^n$$

$$-2y = \sum_{n=0}^{\infty} (-2) a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{\boxed{n-2}}$$ We want x^n

Let $K = n-2$ $n = K+2$

$$y'' = \sum (K+2)(K+1) a_{K+2} x^K$$

$$= \sum_{K=0}^{\infty} (K+2)(K+1) a_{K+2} x^K$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+1} x^n$$

$$y'' - 2xy' - 2y = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} -2n a_n x^n + \sum_{n=0}^{\infty} -2a_n x^n$$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} - 2n a_n - 2a_n\right) x^n$$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} - 2(n+1) a_n\right) x^n = 0$$

Hence

(b) recurrence

$(n+2)(n+1) a_{n+2} - 2(n+1) a_n = 0$ for all $n \geq 0$

$(n+2)(n+1)\, a_{n+2} = \dfrac{2(n+1)\, a_n}{(n+2)(n+1)}$ (since $(n+2)(n+1) \neq 0$)

$a_{n+2} = \dfrac{2a_n}{n+2}$ for $n \geq 0$

(c) $a_2 = a_0$

$a_4 = 1/2\, a_0$

$a_6 = (1/3)(1/2) a_0$

$a_8 = (1/4)(1/3)(1/2) a_0$

$a(2n) = \dfrac{1}{(n)(n-1)\ldots(2)} a_0 = \dfrac{a_0}{n!}$ for $n \geq 0$

$a_3 = (2/3)(a_1)$

$a_5 = (2/5)(2/3) \cdot a_1$

$a_7 = (2/7)(2/5)(2/3) a_1$

$$a(2n+1) = \frac{2}{(2n+1)(2n-1)\ldots(3)} a_1$$

n	$2n+1$	2^n
0	1	1
1	3	2^1
2	5	2^2
3	7	2^3
4	9	2^4

d) write general soln using the general formula for the coefficients

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \ldots) + (a_1 x + a_3 x^3 + a_5 x^5 + \ldots)$$

$$= \sum_{n=0}^{\infty} (a_{2n} x^{2n}) + \sum_{n=0}^{\infty} (a_{2n+1} x^{2n+1})$$

$$= \sum_{n=0}^{\infty} \frac{a_0}{n!} x^{2n} + \sum_{n=0}^{\infty} \frac{2a_1 x^{2n+1}}{(2n+1)(2n-1)\ldots(3)(1)}$$

$$y = a_0\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right) + a_1\left(\sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{(2n+1)(2n-1)\ldots(3)}\right)$$

where a_0, a_1 are any constants

note: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{{x^2}^n}{n!}$$

(Bonus) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = e^{x^2}$