Review of Power Series

The Geometric Series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

converges for $|x| < 1$ &

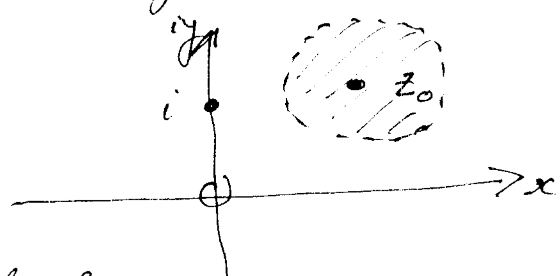
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

Definition Let $f(z)$ be a complex function. $f(z)$ is analytic at $z = z_0$ if it has a power series expansion

$$f(z) = a_0 + a_1 (z - z_0)^1 + a_2 (z - z_0)^2 + a_3 (z - z_0)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for } |z - z_0| < R$$

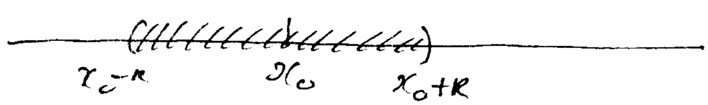
where R is some positive number.



Usually we will restrict to real functions

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $|x - x_0| < R$, some $R > 0$. Hence $a_n \in \mathbb{R}$ for $n \geq 0$.



<u>Theorem</u> Let $R>0$

(1) If $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for $|x-x_0|<R$, then

(a) $a_0 = f(x_0)$, $a_n = \dfrac{f^{(n)}(x_0)}{n!}$ for $n \geq 1$.

(b) $$f'(x) = \sum_{n=0}^{\infty} a_n \, n (x-x_0)^{n-1}$$
$$= \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \quad \text{for } |x-x_0| < R.$$

(c) $$\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1} + C \quad \text{for } |x-x_0| < R.$$

(2) Any rational function
$$R(x) = \frac{p(x)}{q(x)} \quad (p(x), q(x) \text{ polynomials})$$
is analytic at every point x_0 where $q(x_0) \neq 0$.

(3) The functions
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
are ~~any~~ analytic everywhere.

Example

Let $R(x) = \dfrac{x^2 - x - 2}{x^2 + 3x + 2}$. Where is $R(x)$ analytic?

$x^2 + 3x + 2 = (x+1)(x+2) = 0$ when $x = -1, -2$.

So $R(x)$ is analytic everywhere except at $x = -1, -2$.

The points $x = -1, -2$ are called singularities.

$$R(x) = \frac{x^2 - x - 2}{x^2 + 3x + 2}$$

$$= \frac{(x-2)(x+1)}{(x+1)(x+2)}$$

$$= \frac{x-2}{x+2} \qquad (\text{if } x \neq -1, -2).$$

So $x = -1$ is called a removable singularity.