(21) $y'' - xy' + 4y = 0$.

$p(x) = -x$, $q(x) = 4$ are analytic everywhere & $x=0$ is an ordinary point.

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$4y = \sum_{n=0}^{\infty} 4 a_n x^n$$

$$-xy' = \sum_{n=0}^{\infty} (-n a_n)\, x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n$$

$$y'' - xy' + 4y = \sum_{n=0}^{\infty} \left((n+2)(n+1)\, a_{n+2} - n a_n + 4 a_n\right) x^n = 0$$

So $(n+2)(n+1)\, a_{n+2} + (-n+4) a_n = 0$ for all $n \geq 0$,

& $$a_{n+2} = \frac{(n-4)\, a_n}{(n+1)(n+2)} \quad \text{for } n \geq 0.$$

$$a_2 = \frac{-4}{2} a_0 = -2a_0$$

$$a_3 = \frac{(-3)\,a_1}{3\cdot 2}$$

$$a_4 = \frac{-2\,a_2}{4\cdot 3} = -\frac{1}{6} a_2 = \tfrac{1}{3} a_0$$

$$a_5 = \frac{(-1)\,a_3}{5\cdot 4} = \frac{(-3)(-1)}{5\cdot 4\cdot 3\cdot 2}\, a_1$$

$$a_6 = 0$$

$$a_7 = \frac{1\cdot a_5}{7\cdot 6} = \frac{(-3)(-1)(1)\,a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$a_8 = 0$$

$$a_9 = \frac{(+3)\,a_7}{9\cdot 8} = \frac{(-3)(-1)(1)(3)\,a_1}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

We see that

$a_2 = -2a_0$, $a_4 = \tfrac{1}{3} a_0$, and $a_{2n} = 0$ for $n \geqslant 3$.

Also

$$a_{2n+1} = \frac{(-3)(-1)(1)\cdots(2n-5)}{(2n+1)!}\, a_1 \quad \text{for } n \geqslant 0.$$

Therefore

$$y = \sum_{n=0}^{\infty} a_{2n}\, x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}\, x^{2n+1}$$

$$= \left(a_0 - 2a_0 x^2 + \tfrac{1}{3} a_0 x^4\right) + \sum_{n=0}^{\infty} \frac{(-3)(-1)\cdots(2n-5)\,a_1 x^{2n+1}}{(2n+1)!}$$

$$= a_0\left(1 - 2x^2 + \tfrac{1}{3} x^4\right) + a_1 \sum_{n=0}^{\infty} \frac{(-3)(-1)\cdots(2n-5)\,x^{2n+1}}{(2n+1)!}$$

where a_0, a_1 are any constants.