8.4 Equations with analytic coefficients

Theorem Suppose x_0 is an ordinary point of the DE

(*) $y'' + p(x)y' + q(x)y = 0$.

Then (*) has two linearly independent solutions that are analytic at x_0. Moreover the radius of convergence of ~~the general soln~~ any soln

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

is at least the distance of x_0 to the nearest singular point (real or complex).

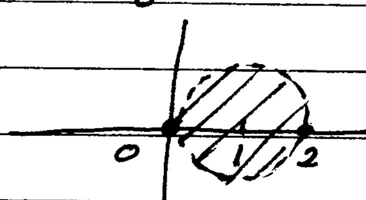
Example (#9)

$$(x^2-2x)y'' + 2y = 0, \qquad x_0 = 1.$$

$$y'' + \frac{2}{x^2-2x} y = 0.$$

$p(x) = 0 \qquad q(x) = \dfrac{2}{x(x-2)}$ is not analytic at $x=0, 2$.

Any soln near $x_0 = 1$ has radius of convergence $\rho \geq 1$.



$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=0}^{\infty} a_n (x-1)^{n-1} n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$x^2 - 2x = (x-1)^2 - 1$$

$$(x^2-2x)y'' = \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n - \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

$$= \sum_{n=0}^{\infty} \left[n(n-1)a_n - (n+2)(n+1)a_{n+2} \right] (x-1)^n$$

$$(x^2-2x)y'' + 2y = \sum_{n=0}^{\infty} \left[(n^2-n+2)a_n - (n+2)(n+1)a_{n+2} \right] (x-1)^n$$

$$(n+2)(n+1)a_{n+2} = (n^2-n+2)a_n \qquad n \geq 0$$

$$a_{n+2} = \frac{(n^2-n+2)\,a_n}{(n+2)(n+1)}$$

$$a_2 = a_0$$

$$a_3 = \frac{2}{3\cdot 2}a_1 = \frac{1}{3}a_1$$

$$a_4 = \frac{(4-2+2)}{(4)(3)}a_2 = \frac{1}{3}a_2 = \frac{1}{3}a_0$$

Hence

$$y = a_0 + a_1(x-1) + a_0(x-1)^2 + \tfrac{1}{3}(x-1)^3 a_1 + \tfrac{1}{3}(x-1)^4 a_0 + \cdots$$

$$= a_0\left(1 + (x-1)^2 + \cdots\right)$$

$$+ a_1\left((x-1) + \tfrac{1}{3}(x-1)^3 + \cdots\right)$$

for any constants a_0, a_1 & $|x-1| < 1$.

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$$(x^2+1)y'' - e^x y' + y = 0 \qquad y(0)=1,\ y'(0)\ 1.$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(x^2+1)y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$-e^x y' = \left(-1 - x - \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots)$$

$$= -a_1 + (-2a_2 - a_1)x + \left(-\frac{a_1}{2} - 2a_2 - 3a_3\right)x^2 + \cdots$$

$$(x^2+1)y'' = (x^2+1)(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots)$$

$$= 2a_2 + 6a_3 x + (2a_2 + 12a_4)x^2 + \cdots$$

x^0: $a_0 - a_1 + 2a_2 = 0$

$$2a_2 = 0 \qquad a_2 = 0$$

x^1: $a_1 - a_1 - 2a_2 + 6a_3 = 0$

$$a_3 = 0$$

x^2: $a_2 - \frac{a_1}{2} - 2a_2 - 3a_3 + 2a_2 + 12a_4 = 0$

$$-1/2 + 12a_4 = 0 \qquad a_4 = \frac{1}{24}$$

x^3: $a_3 - 1/6 - 4a_4 + (20a_5) = 0$

$$2a_5 = \frac{1}{6} + 4a_4 = \frac{1}{6} + \frac{1}{6} = 1/3$$

$$a_5 = \frac{1}{60}$$

$$y = 1 + x + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \cdots$$

#25 $(1+x^2)y'' - xy' + y = e^{-x}$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$-xy' = -a_1 x - 2a_2 x^2 - 3a_3 x^3 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + \cdots$$

$$x^2 y'' = 2a_2 x^2 + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$$(1+x^2)y'' - xy' + y$$

$$= (a_0 + 2a_2) + (\cancel{a_3}\, 6a_3)x + \cdots$$

$$= 1 - x + \cdots$$

$$2a_2 = 1 - a_0$$

$$6a_3 = -1$$

$$a_3 = -1/6$$

$$y = a_0 + a_1 x + \left(\frac{1}{2} - \frac{a_0}{2}\right)x^2 - \frac{1}{6}x^3 + \cdots,$$

where a_0, a_1 are any constants.