8.5 Cauchy-Euler Equations

(*) $ax^2y'' + bxy' + cy = 0, \quad x > 0.$

$\Leftrightarrow \quad y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = 0$

$\boxed{p(x) = \frac{p_0}{x}, \; q(x) = \frac{q_0}{x^2}}$

$x = 0$ is a singular point.

We try

$$y = x^r$$
$$y' = rx^{r-1}$$
$$xy' = rx^r$$
$$y'' = r(r-1)x^{r-2}$$
$$x^2y'' = r(r-1)x^r$$

$$ax^2y'' + bxy' + cy = x^r(a(r)(r-1) + br + c) = 0$$

So if

$$ar(r-1) + br + c = 0 \quad \text{(Auxiliary Eqn).}$$

Then x^r is a soln to (*).

Case 1 A.E. has two distinct real roots r_1, r_2.

Then $y_1 = x^{r_1}$, $y_2 = x^{r_2}$ are linearly indep't. solns to (*).

Case 2 A.E. has complex roots

$$r = \alpha \pm i\beta.$$

$$x^{\alpha + i\beta} = e^{\ln x(\alpha + i\beta)} = (e^{\ln x})^{\alpha} e^{i\beta \ln x}$$
$$= x^{\alpha}(\cos(\beta \ln x) + i \sin(\beta \ln x)).$$

Then

$y_1 = x^{\alpha}\cos(\beta \ln x)$, $y_2 = x^{\alpha}\sin(\beta \ln x)$ are