has two indept. solns to (*) (18)

CASE 3

AE: $a(r)(r-1)+br+c = ar^2+(b-a)r+c=0$

has a double root. $r = \dfrac{-(b-a) \pm \sqrt{\Delta}}{2a}$

$\Delta = (b-a)^2-4ac=0$ & double root is $r=r_0=\dfrac{-(b-a)}{2a}$.

$y_1 = x^{r_0}$, $y_2=(\ln x)\,x^{r_0}$

are 2 linearly indept. solns $(x>0)$.

Let
$$y = (\ln x)\,x^r$$
$$y' = (\ln x)\,rx^{r-1} + x^{r-1}$$
$$y'' = (\ln x)\,r(r-1)x^{r-2} + rx^{r-2} + (r-1)x^{r-2}$$
$$= (\ln x)\,r(r-1)x^{r-2} + (2r-1)x^{r-2}$$
$$y = (\ln x)\,x^r$$
$$xy' = \cancel{(\ln x)\,x^r}\ x^r(r\ln x+1)$$
$$x^2y'' = x^r(r(r-1)\ln x + (2r-1))$$

$$ax^2y''+bxy'+cy$$
$$= x^r\big((ar(r-1)+br+c)\ln x + a(2r-1)+b\big)$$

$a r_0(r_0-1)+br_0+c=0$

$a(2r_0-1)+b = 2r_0a-a+b = -b+a-a+b=0$

So $y_2=(\ln x)\,x^{r_0}$ is a soln.