(5) ~~Let F, G be as before~~

~~Define $L: F \to G$ by~~

Let F = set of functions $f(x)$ where $f''(x)$ exists.

Let G = set of functions.

Define $L: F \longrightarrow G$ by

$L(f) = f'' + (\sin x) f' + (x^2+1) f$.

Let $f_1, f_2 \in F$ & $c \in \mathbb{R}$.

$L(f_1 + f_2) = (f_1 + f_2)'' + (\sin x)(f_1 + f_2)' + (x^2+1)(f_1 + f_2)$

$= f_1'' + f_2'' + (\sin x)(f_1' + f_2') + (x^2+1)(f_1 + f_2)$

$= f_1'' + (\sin x) f_1' + (x^2+1) f_1$
$+ f_2'' + (\sin x) f_2' + (x^2+1) f_2$
$= L(f_1) + L(f_2)$.

$L(cf_1) = (cf_1)'' + (\sin x)(cf_1)' + (x^2+1)(cf_1)$
$= cf_1'' + (\sin x)(c)(f_1') + (x^2+1)(cf_1)$
$= c(f_1'' + (\sin x) f_1' + (x^2+1) f_1)$
$= c\, L(f_1)$

Hence L is linear.

(6) Let $a_1(x), a_2(x), a_3(x)$ be continuous functions

Define $L: F \longrightarrow G$ by

$L(f) = a_1(x) f'' + a_2(x) f' + a_3(x) f$

Then L is a linear transformation.

(7) Define $L: F \to G$ by $L(y) = y\, y'$.

L is not linear. Eg Let $f_1(x) = x$ Let $f_2(x) = 1$

$L(f_1 + f_2) = L(x+1) = (x+1)(x+1)'$

$= (x+1) \cdot (1)$

$= x + 1$

$L(f_1) = x(x)' = x \quad L(f_2) = 1(1)' = 0$