Linear Independance of Functions

The functions $f_1(x), f_2(x), \ldots, f_m(x)$ are linearly independent on an interval I if $c_1, c_2, \ldots, c_m$ are constants and

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0 \quad \text{for all } x \in I$$

implies $c_1 = c_2 = \cdots c_m = 0$.

Otherwise the functions are linearly dependent on the interval I.

Example Show

$$f_1(x) = \sin x, \; f_2(x) = \cos x, \; f_3(x) = e^x$$

are linearly independent on $(-\infty, \infty)$.

Suppose c_1, c_2, c_3 are constants and

$$c_1 \sin x + c_2 \cos x + c_3 e^x = 0 \quad \text{for all } x.$$

Let $x = 0$ do $\boxed{c_2 + c_3 = 0.}$ (1)

Take derivative &

$$c_1 \cos x - c_2 \sin x + c_3 e^x = 0 \quad \text{for all } x.$$

Let $x = 0$ do $\boxed{c_1 + c_3 = 0}$ (2)

Take derivative again &

$$-c_1 \sin x - c_2 \cos x + c_3 e^x = 0 \quad \text{for all } x.$$

Let $x = 0$ do $\boxed{-c_2 + c_3 = 0}$ (3)

(1) + (3) $\Rightarrow$ $2c_3 = 0$ & $c_3 = 0$.

(1) $\Rightarrow$ $c_2 = 0$, (2) $\Rightarrow$ $c_1 = 0$ & hence

$$c_1 = c_2 = c_3 = 0.$$

Hence the three functions are linearly independent on $(-\infty, \infty)$.