Theorem Let $p_{n-1}(x), \ldots, p_1(x), p_0(x)$ be CTS on (a,b). Suppose $y_1, y_2, \ldots, y_n$ are solutions of

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0.$$

TFAE (the following are equivalent)

(1) $y_1, y_2, \ldots, y_n$ are linearly independent on (a,b)

(2) $W[y_1, y_2, \ldots, y_n](x_0)$ is non-zero at some point $x_0 \in (a,b)$.

(3) The Wronskian $W[y_1, y_2, \ldots, y_n](x)$ is never zero on (a,b).

Example Show that $\{1, x, x^3\}$ are a fundamental set of solutions to the DE

$$y''' - \frac{1}{x}y'' = 0 \qquad \text{for } x>0. \quad (\text{ie } (0,\infty)).$$

$y_1 = 1 \qquad y_2 = x \qquad y_3 = x^3$

$y_1' = 0 \qquad y_2' = 1 \qquad y_3' = 3x^2$

$y_1'' = 0 \qquad y_2'' = 0 \qquad y_3'' = 6x$

$y_1''' = 0 \qquad y_2''' = 0 \qquad y_3''' = 6$

$y_1''' - \frac{1}{x}y_1'' = 0$ & y_1 is a solution.

$y_2''' - \frac{1}{x}y_2'' = 0$ & y_2 is a solution.

$y_3''' - \frac{1}{x}y_3'' = 6 - \frac{1}{x}6x = 6-6 = 0$ & y_3 is a solution.

$$W[y_1, y_2, y_3] = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 6x > 0 \quad \text{for } x>0.$$

So $\{1, x, x^3\}$ is a fundamental ~~set~~ of solutions.