NON HOMOGENEOUS LINEAR DEs

Theorem Let $p_{n-1}(x), \ldots, p_1(x), p_0(x)$ be CTS on (a,b). Suppose $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions to

$$(*) \quad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0.$$

Suppose y_p is a particular solution of the nonhomogeneous DE

$$(**) \quad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x)$$

where $g(x)$ is CTS on (a,b).

Then the general solution of $(**)$ is given by

$$y = y_p + c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

where $c_1, c_2, \ldots, c_n$ are any constants.

Proof: Assume the hypotheses of the theorem. Let L be the linear operator

$$L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y.$$

We are given that

$$L[y_p] = g(x).$$

Suppose $\phi(x)$ is any solution of $(**)$; ie.

$$L[\phi] = g(x).$$

Then

$$\begin{aligned}
L[\phi - y_p] &= L[\phi + (-1)y_p] \\
&= L[\phi] + L[(-1)y_p] && \text{(since } L \text{ is linear)} \\
&= g(x) + (-1)\,L[y_p] && \text{(since } L \text{ is linear)} \\
&= g(x) + (-1)\,g(x) \\
&= 0.
\end{aligned}$$