Hence $y = \phi - y_p$ is a soln. of (*).
Thus
$$\phi - y_p = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$
for some constants $c_1, c_2, \ldots, c_n$ &
$$\phi = y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$
Conversely suppose $y = y_p + c_1 y_1 + \cdots + c_n y_n$. Then
$$\begin{aligned} L[y] &= L[y_p + (c_1 y_1 + \cdots + c_n y_n)] \\ &= L[y_p] + L[c_1 y_1 + \cdots + c_n y_n] \\ &= \phi + 0 \\ &= \phi \end{aligned}$$
& y is a solution to (**). $\square$

<u>Example</u> (#20)

(a) Find the general solution of the nonhomogeneous DE
$$xy''' - y'' = -2 \Leftrightarrow y''' - \tfrac{1}{x} y'' = \tfrac{-2}{x} \quad (\text{assume } x > 0).$$
given that $y_p = x^2$ is a particular solution & $\{1, x, x^3\}$ is a fundamental set of solutions for the corresponding homogeneous DE. We check y_p:
$$y_p' = 2x, \ y_p'' = 2, \ y_p''' = 0,$$
$$x y_p''' - y_p'' = -2 \ \&$$
y_p is a particular solution. Thus the general solution is
$$y = x^2 + c_1 + c_2 x + c_3 x^3 \quad \text{where } c_1, c_2, c_3$$
are any constants.

(b) Solve the IVP $xy''' - y'' = -2$, $y(1) = 2$, $y'(1) = -1$, $y''(1) = -4$.
$$\Leftrightarrow y''' - \tfrac{1}{x} y'' = \tfrac{-2}{x} \quad \text{for } x > 0.$$

$$y(1) = 2 + c_1 + c_2 + c_3 = 2$$
$$y' = 2x + c_2 + 3x^2 c_3$$
$$y'(1) = 2 + c_2 + 3c_3 = -1$$